DOUBLE OUTER INDEPENDENT EDGE DOMINATION IN GRAPHS

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ABSTRACT:

We introduce the concept of double outer independent edge domination in graph. The set S of edges of a graph G, in which every edge of graph G is dominated by at least two edges of S, and the set of edges $E(G)-S$ is free means no two edges are adjacent, is called double outer independent edge dominating set of a graph G, represented by γ'_{doi} - set of G. The least number of edges contained in this set is called γ'_{doi} - number of G. Initially, some straight forward and inequality results were given. Also we describe the result for extremal graphs. Further, the effect of deleting the edges is also discussed. At the end, Nordhaus – Gaddum type inequalities were obtained.

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INTRODUCTION:

Let $G=(V,E)$ be the graph. The number of vertices and edges of G denoted by p and q respectively. The graph which has equal number of vertices as G, in which two vertices are adjacent if and only if they are not adjacent in G is called complimentary graph, represented by $\bar{G}$. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices of X and $N(v)\left(N[v]\right)$ denote the open (closed) neighborhoods of a vertex v. $\deg(v)$ is the degree of a vertex v. For any undefined terms in this paper, follow the notations of [2].

For an edge e, $N(e)\left(N[e]\right)$ denote the open (closed) neighborhoods of an edge e. The degree of an edge $e=uv$ is defined by $\deg(e)=\deg(u)+\deg(v)-2$. The edge $e=uv$ is called an end edge, if either $\deg(u)=1$ or $\deg(v)=1$, its neighbor is called support edge. $\delta'(G)\left(\Delta'(G)\right)$ denote the minimum (maximum) edge degree among the edges of G. The edge independence number $\alpha'(G)$, is the maximum cardinality of independent set of edges of G.

A set $D\subseteq V(G)$ is said to dominating set of G, if every vertex in $V-D$ is adjacent to some vertex in D. The domination number of G, denoted by $\gamma(G)$ is the cardinality of a minimum dominating set of G. The concept of domination with its many variations is now well studied in graph theory, ([3], [4] and [5]).

An edge set $S\subseteq E(G)$ is said to be an edge dominating set of G, if every edge in $E-S$ is adjacent to some edges in S. The edge domination number of G, denoted by $\gamma'(G)$ is the cardinality of minimum edge dominating set of G. (see [1]).

An edge dominating set S of graph is called double edge dominating set, if every edge of graph is dominated by not less than two edges in S. The minimum number of edges in such set is called double edge domination number of graph G, which is represented by $\gamma'_d(G)$. (see [6], [7] and [8]).

The set S of edges of a graph G, in which every edge of graph G is dominated by at least two edges of S, and the set of edges $E(G)-S$ is free means no two edges are adjacent, is called double outer independent edge dominating set of a graph G, represented by γ'_{doi} - set of G. The least number of edges contained in this set is called γ'_{doi} number of G. Initially, some straight forward and inequality results were given. Also we describe the result for extremal graphs. Further, the effect of deleting the edges is also discussed. At the end, Nordhaus – Gaddum type inequalities were obtained.

RESULTS:

Initially, we give the following Observations which are straight forward.

Observation 1: Each end edge of a graph G is in each double outer independent edge dominating set of G.

Observation 2: Each support edge of a graph G is in each double outer independent edge dominating set of G.

Next result double outer independent edge dominating set interms of minimum edge degree of G.

Theorem 1: For any connected graph G, $\delta'(G) \le \gamma'_{doi}(G)$.
Proof: Let $S = \{e_1, e_2, ..., e_n\}$ be the outer independent edge dominating set of G. If $S = E(G)$, then the result is obvious. Now assume that $S \ne E(G)$, then there exists at least one edge $e_x \notin S$. Since the sub graph $\langle E(G)-S \rangle$ is independent, $N(e_x) \in S$. Further there exists at least one edge $\{e\}$ such that $|e| = \delta'(G)$, it follows that $|e| = S$. Therefore, $\delta'(G) \le \gamma'_{doi}(G)$.

In the following Theorems, we give characterization for double outer independent edge domination number.

Theorem 2: For any connected (p,q) - graph G, $\gamma'_{doi}(G) = q$ if and only if every edge of G is an end edge or support edge.
Proof: The sufficiency is true by Observations 1 and 2. Now assume that, some edge of a graph say e_x is neither an end edge nor a support edge. Thus e_x has at least two neighbors in S. Moreover each of these neighbors has a neighbor different from e_x. Since e_x is not a support edge. Clearly, it follows that $E(G)-\{e_x\}$ is a double outer independent edge dominating set of the graph G. Hence $\gamma'_{doi}(G) \le q-1 < q$.

Theorem 3: For any connected (p,q) - graph G with $q \ge 3$ edges, $\gamma'_{doi}(G) = q-1$ if and only if at least one edge of G is neither an end edge nor a support edge, and the sub graph of G induced by the edges which are neither end edges nor support edges is a complete graph or a path on three edges such that the edge incident with central vertex has exactly two neighbors in graph G.
Proof: Let G be a graph such that its sub graph induced by the edges which are neither end edges nor support edges is a complete graph or a path on three edges such that the central edge has exactly two neighbors in the graph G. First assume that, it is a path P_4, say $e_a e_b e_c$. It is not difficult to observe that $E(G)-\{e_b\}$ is a double outer independent edge dominating set of the graph G. Thus $\gamma'_{doi}(G) \le q-1$. At present consider S be some γ'_{doi}-set of G. From Observations 1 and 2, all end edges and support edges belong to the set S. Moreover, the edge e_b has to be dominated twice, thus at least two of the edges e_a, e_b and e_c included in the set S. Consequently, $\gamma'_{doi}(G) \ge q-1$. Now assume that the sub graph of G induced by the edges which are neither end edges nor

support edges is a complete graph. Let e_x be any edge of this sub graph. Let us observe that $E(G)-\{e_x\}$ is a double outer independent edge dominating set of graph G. Thus $\gamma^{'}_{doi}(G)\leq q-1$. Now, let S be any $\gamma^{'}_{doi}$-set of G. By Observations 1 and 2, all end edges and support edges belong to the set S. Moreover, since $E(G)-S$ is independent, at most one of the remaining edges is not the part of the set S. Consequently, $\gamma^{'}_{doi}(G)\geq q-1$. Hence, at present we conclude that $\gamma^{'}_{doi}(G)=q-1$.

Now imagine that for certain graph G, we get $\gamma^{'}_{doi}(G)=q-1$. Suppose that the sub graph of G induced by the edges which are neither end edges nor support edges, say G_1, is neither a complete graph nor a path on three edges such that the edge incident with central vertex has exactly two neighbors in the graph G. Thus $|E(G_1)|\geq 2$. Let us observe that there exists two non adjacent edges of G_1, say e_x and e_y, such that no common neighbor of e_x and e_y has edge degree two in the graph G. Clearly, it follows that $E(G)-\{e_x,e_y\}$ is a double outer independent edge dominating set of the graph G. Therefore, $\gamma^{'}_{doi}(G)\leq q-2<q-1$, a contradiction.

In the following Theorem, relation between $\gamma^{'}_{doi}(T)$ interms of $\alpha^{'}(T)$ and number of support edges of trees.

Theorem 4: For any connected tree T of size at least three, $\gamma^{'}_{doi}(T)\leq \alpha^{'}(T)+m$, where m is the number of support edges.

Proof: Let q be the numeral of edges of tree T. We begin by induction on q. If $diam(T)=2$, resulting T is a star $K_{1,n}$. We must have $\gamma^{'}_{doi}(K_{1,n})=q-1>1+q=\alpha^{'}(K_{1,n})+m$. Now, imagine that $diam(T)$ $=3$. Thus T is a double star. We have $\gamma^{'}_{doi}(T)=q=q-1+1=\alpha^{'}(T)+m$.

At present presume that $diam(T)\geq 4$. Therefore, the size q of the tree T is at least four. Here the result is obtained by induction on the number of edges q. Assume that the Theorem is true for every tree $T^{'}$ of size $q^{'}<q$.

Initially, imagine that few support edge of T, assume e_x is adjacent to more than one edge. Let e_y be an end edge neighbor to e_x. Consider $T^{'}=T-e_y$. We have $m^{'}=m$. Let $S^{'}$ be any $\gamma^{'}_{doi}$- set of T. From Observation 2, it holds $e_x\in S^{'}$. There is no complication to observe that $S^{'}\cup\{e_y\}$ is the double outer independent edge dominating set of the tree T. Thus $\gamma^{'}_{doi}(T)\leq \gamma^{'}_{doi}(T^{'})+1$. Now to inspect that there occurs a greatest edge set of $T^{'}$ that does not contain the edge e_x. Consider $A^{'}$ to be such a set. There is no complication to inspect that $S^{'}\cup\{e_y\}$ is an independent edge set of tree T. Thus $\alpha^{'}(T)\geq\alpha^{'}(T^{'})+1$. Clearly, it follows that $\gamma^{'}_{doi}(T)\leq\gamma^{'}_{doi}(T^{'})+1\leq\alpha^{'}(T^{'})+m^{'}+1$ $=\alpha^{'}(T^{'})+m+1\leq\alpha^{'}(T)+m$. Hence, let us imagine that all edges of T adjacent to end edges are weak. At present the tree T is established at an edge e_x incident with the vertex x of greatest distance called diameter of tree. Consider e_t be an end edge at greatest difference from e_r, e_v be the rear of e_t, e_u be the rear of e_v and e_w be the rear of e_u in the established tree. By T_{e_x} we indicate the sub tree produced from an edge e_x and its descendents in the established tree T.

Presume now that $d(e_u)\geq 3$. Let $T^{'}=T-T_{e_v}$. We have $m^{'}=m-1$. Consider $S^{'}$ as some $\gamma^{'}_{doi}$- set of $T^{'}$. Evidently, $S^{'}\cup\{e_v,e_t\}$ is a double outer independent edge dominating set of tree T. Thus $\gamma^{'}_{doi}(T)$ $\leq\gamma^{'}_{doi}(T^{'})+2$. At present, consider $A^{'}$ be greatest independent edge set of tree $T^{'}$. There is no complication to inspect that $S^{'}\cup\{e_t\}$ is an independent edge set of tree T. Thus $\alpha^{'}(T)\geq\alpha^{'}(T^{'})+1$. We now get $\gamma^{'}_{doi}(T)\leq\gamma^{'}_{doi}(T^{'})+2\leq\alpha^{'}(T^{'})+\ m^{'}+2\leq\alpha^{'}(T)+m$.

At present, presume $d(e_u)=2$. Initially, imagine that there is a frontal of e_w different from e_u, assume e_k, namely the distance from e_w to the farthest edge of T_{e_k} is one or three. It satisfies to take into consideration only the possibilities, at the same time T_{e_k} is a trail P_4, or e_k is an end edge. Let $T' = T - T_{e_u}$. We have $m' = m - 1$. Now to inspect that, in that place occurs a γ'_{doi}- set of T' that contains the edge e_w. Let S' be a similar set. There is no complication to notice that $S' \cup \{e_v, e_t\}$ is a double outer independent edge dominating set of the tree T. Thus $\gamma'_{doi}(T) \le \gamma'_{doi}(T') + 2$. At present, consider A' be maximum independent edge set of tree T'. Obviously, $S' \cup \{e_t\}$ represents the set of edges which are not adjacent to each other. Thus $\alpha'(T) \ge \alpha'(T') + 1$. We now get $\gamma'_{doi}(T) \le \gamma'_{doi}(T')$ $+2 \le \alpha'(T') + m' + 2 \le \alpha'(T) + m$.

At present, presume that for every child of e_w other than e_u, say e_k, the interval of e_w to the most farthest edge of T_{e_k} is two. It satisfies to take into consideration only the possibilities, at the same time e_k is a non end edge adjacent to minimum two edges. Let $T' = T - T_{e_v}$. We have $m' = m$. Let S' be any γ'_{doi}- set. By Observations 1 and 2, we have $e_u, e_k, e_w \in S'$. Let us observe that $S' - (\{e_u\} \cup \{e_v, e_t\})$ is a double outer independent edge dominating set of the tree T. Consequently, $\gamma'_{doi}(T) \le \gamma'_{doi}(T') + 1$. At present consider A' be the greatest independent edge set of tree T'. There is no complication to inspect that $S' \cup \{e_t\}$ is a set of edges which are not adjacent to each other in tree T. Consequently, $\alpha'(T) \ge \alpha'(T') + 1$. We now get $\gamma'_{doi}(T) \le \gamma'_{doi}(T') + 1 \le \alpha'(T') + m' + 1 \le \alpha'(T) + m$.

Next result shows the effect of deleting an edge of a graph on γ'_{doi} number.

Theorem 5: For any connected graph G of size at least three, $\gamma'_{doi}(G) - 2 \le \gamma'_{doi}(G - e) \le \gamma'_{doi}(G) + \deg(e) - 1$.

Proof: Let S be any γ'_{doi}- set of the graph G. If $e \notin S$, then it is easy to see that S is a double outer independent edge dominating set of the graph $G - e$. Assume now that $e \in S$. Let $e_1, e_2, \ldots, e_{d(e)}$ be the neighbor of e. Let $i \in \{1, 2, \ldots, d(e)\}$. Let e_{x_i} be the neighbor of e_i different from e. If $e_i \in S$, then let e_{u_i} mean e_{x_i}, otherwise let it mean e_i. Let us observe that $S \cup \{e_{u_1}, e_{u_2}, \ldots, e_{u_{d(e)}}\} - \{e\}$ is a double outer independent edge dominating set of the graph G. Thus $\gamma'_{doi}(G - e) \le \left|S \cup \{e_{u_1}, e_{u_2}, \ldots, e_{u_{d(e)}}\} - \{e\}\right| \le |S| + d(e) - 1$ $= \gamma'_{doi}(G) + d(e) - 1$. Further, let S' be any γ'_{doi}- set of $G - e$. If some edge of $N(e)$ belongs to the edge set S', then it is easy to see that $S' \cup \{e\}$ is a double outer independent edge dominating set of the graph G. Assume now that, no edge of $N(e)$ belongs to the edge set S'. Let e_x be any neighbor of e in G. Clearly, it follows that $S' \cup \{e, e_x\}$ is a double outer independent set of G. Therefore, $\gamma'_{doi}(G)$ $\le \gamma'_{doi}(G - e) + 2$ and hence $\gamma'_{doi}(G) - 2 \le \gamma'_{doi}(G - e)$.

We now give Nordhaus – Gaddum type inequalities in the following Theorems.

Theorem 6: For any connected (p,q) - graph G, $q - 1 \le \gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) \le 2q$.

Proof: Let S be any γ'_{doi}- set of G. Since $E(G) - S$ is independent edge set, the edges of $E(G) - S$ forms a clique in $\bar{G}$. Let $\bar{S}$ be any γ'_{doi}- set of $\bar{G}$. The edges of $E(G) - S$ forms a clique in $\bar{G}$, thus at most one of them does not belong to $\bar{S}$ as $E(\bar{G}) - \bar{S}$ is an independent edge set. Therefore, $|\bar{S}| \ge |E(G) - S| - 1$. It follows that, $\gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) \ge |S| + |E(G) - S| - 1 = q - 1$. Obviously, $\gamma'_{doi}(G) \le q$ and γ'_{doi} $(\bar{G}) \le q$. Thus we have, $\gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) \le 2q$.

Theorem 7: For any connected (p,q) - graph G, $\gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) = 2q - 1$ if and only if $G = P_5$.

Proof: Obviously, $\gamma'_{doi}(P_5) + \gamma'_{doi}(\bar{P}_5) = 2\gamma'_{doi}(P_5) - 1 = 2q - 1$. At present, presume any graph G possesses $\gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) = 2q - 1$. This implies that $\gamma'_{doi}(G) = q$ and $\gamma'_{doi}(\bar{G}) = q - 1$. By Theorem 2, every edge of G is an end edge or support edge. Suppose that some edge, say e_x, is simultaneously an end edge or support edge of G. This implicates that the component of G which contains the edge e_x is a complete graph. If $G = P_3$, then $\bar{G}$ has more than one component, a contradiction. Now assume that $G \neq P_3$. Since every component of G has at least two edges, we have $q \geq 4$. In the graph $\bar{G}$, the edge e_x is not an end edge as it has $q-2$ neighbors and $q-2 \geq 2 > 1$. Thus e_x must be a support edge of $\bar{G}$. But each neighbor of e_x in $\bar{G}$ is also adjacent to e_y, so e_x is not adjacent to any end edge and it is not a support edge, a contradiction. We now conclude that, no edge is simultaneously an end edge and support edge neither of graph G nor of $\bar{G}$. Suppose that some edge, say e_x is an end edge of graph G. subsequently G has exactly one neighbor. Since the graph G and $\bar{G}$ has three edges, it is not difficult to verify that G or $\bar{G}$ has more than one component and we do not consider such graphs. Therefore, end edge of G becomes support edge of $\bar{G}$, and every edge of $\bar{G}$ is a support edge of G. Let us observe that for every graph, the number of end edges is greater than or equal to the number of support edges. This implies that exactly half of all edges are end edges of G which are support edges of $\bar{G}$, and the remaining half of edges are support edges of G. Consequently, every support edge is a weak. Moreover, the number $q = |E(G)|$ and $q+2 = |E(\bar{G})|$ is even. Suppose that $q \geq 6$. Thus, in G there are at least three end edges say e_a, e_b and e_c. The support edge adjacent to e_a we denote by e_x. Since every support edge is weak, the edge e_x is not adjacent to any one of the edges e_b and e_c. Thus e_x is adjacent to both edges e_b and e_c in $\bar{G}$, a contradiction. This implies that $q = 4$. Since exactly half of all edges are end edges half are support edges, therefore it is not very difficult to get $G = P_5$.

Theorem 8: For any connected (p,q) - graph G, $\gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) \geq q + l - 2$, where l is the number of end edges.

Proof: Let S and $\bar{S}$ be any γ'_{doi} - set of G and $\bar{G}$ respectively. If some of end edges of G does not belong to the set $\bar{S}$, then $\gamma'_{doi}(\bar{G}) \geq q-2$ and we easily obtain the result. Now assume that all end edges of G belong to the set $\bar{S}$. Since $E(G)-S$ is independent, the edges of $E(G)-S$ forms a clique in $\bar{G}$. Consequently, at most one of them does not belong to $\bar{S}$ as $E(\bar{G})-\bar{S}$ is independent. Therefore, $|\bar{S}| \geq |E(G)-S| + l - 1$. We now get $\gamma'_{doi}(G) + \gamma'_{doi}(\bar{G}) \geq |S| + |E(G)-S| + l - 1 > q + l - 2$.

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