S-METRIC SPACES FIXED POINT RESULTS ON ALTMAN INTEGRAL TYPE MAPPINGS

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ABSTRACT.
In this present article, we establish the concept of φ-weakly commuting self-mappings pairs in S-metric space. with this idea we create a common fixed point theorem of Altman integral type for self-mappings in the context of S-metric space. Example is constructed to support our result.
Keywords: Altman type mapping, fixed point ,common fixed point, self-mapping, φ -weakly commuting self-mapping, weakly compatible.

1. Introduction

Fixed point theory is one of the most dynamic research subject in nonlinear analysis. In the field of metric fixed point theory the first important and significant result was proved by Banach in 1922 for contraction mapping in complete metric space. The well known Banach contraction theorem may be stated as follows."Every contraction mapping of a complete metric space X into itself has a unique fixed point"(Bonsall 1962).

In 1975 Altman [1] introduced generalized contractions proved a fixed point theorem for a single self-mapping in compact metric space satisfying the following contractive condition: Let (X,d) be a metric space and $f : X \to X$ be a function. Then f is called a generalized contraction if for all x,y $\in X$

$$d(Tx, Ty) \leq Q(d(x, y)) \ \forall \ x, y \in X$$

where Q: $[0, \infty) \to [0, \infty)$ is an increasing function satisfies the following conditions:

(1) $0 < Q(t) < t$, $t \in (0, \infty)$; but Q is increasing if Q(0)=0 also Q(t)= t implies and implied by t = 0

(2) $\rho(t) = \frac{t}{t - Q(t)}$ is a decreasing function;

(3) $\int_0^{t_1} \rho(t)dt < +\infty$ for some positive number t_1.

Remark 1.1: By(1) and that Q is increasing we have Q (0) = 0 also Q (t) = t $\Leftrightarrow$ t =0.

2.REVIEW LITERATURE

Common fixed point for Altman type mapping has been discussed by Garbone and Singh [2] and Li and Gu [3] in metric spaces. In 2006, Mustafa and Sims [4] introduced a new structure of generalized metric space called G-metric space. Gu and Ye [5] obtained a common fixed point theorem for Altman integral type mapping in complete G-metric space. Recently, Sedghi et al. [6] initiated the idea of S-metric space as a generalization of G-metric space. While in [7] Sedghi proved fixed point theorems for implicit relation in complete S-metric space. In this paper, we derive a common fixed point Altman integral type mapping for two pairs of ϕ-weakly commuting self-mappings in complete S-metric space. We begin with the following definitions and results in the framework of S-metric space which can be found in [6, 7].common fixed points for non-continuous non self mappings on non-Metric space by G.Jungek [8]A Fixed point theorem for mappings satisfying contractive condition of Integral in (2002).

2. PRELIMINARIES:

Definition 2.1 Let X be a non-empty set. An S-metric is a function $S: X \times X \times X \to [0, \infty)$ satisfying the following conditions for all $x, y, z, a \in X$

S_1) $S(x, y, z) = 0$ if and only if $x = y = z$;

S_2) $S(x, y, z) \leq S(x, a, a) + S(y, a, a) + S(z, a, a)$.

The pair (X, S) is called S-metric space.

Example 2.2. Let X=[0,1],define $S : X \times X \times X \to R^+$ defined by

$S(x, y, z) = (|x - y| + |x - z| + |y - z|)^2$ for all $x, y, z \in X$ .Then (X, S) is a complete S-metric space.

Definition 2.3. Let (X, S) be an S-metric space. A sequence $\{x_n\}$ in X is called

(1) Converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. We write $x_n \to x$ for brevity.

(2)Cauchy sequence if for $\in > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \in$.

(3) to be complete if every Cauchy sequence in X converges in.

Lemma 2.4. Let (X,S) be a S-Metric space then Limit of the convergent sequence in S-metric space is unique and $S(x,x,y)=S(y,y,x)$ for all $x, y \in X$

Now we introduce the concept of ϕ-weakly commuting pairs of self-mappings in S-metric space as follows:

Definition 2.5. A pair of self-mappings (S, T) on S-metric space is called φ -weakly commuting. If there exist a continuous function $\phi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$ such that $S(STx, STx, TSx) \leq \varphi(S(Sx, Sx, Tx))\ \forall x \in X$.

Example 2.6. Let $X = [0, \infty)$, $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Let $S, T : X \to X$ are defined by $Sx = \frac{x}{4}$ and $Tx = \frac{x}{2}$ then

$$S(STx, STx, TSx) = S(\frac{x}{8}, \frac{x}{8}, \frac{x}{8}) \leq \frac{1}{2}\frac{3}{4}x = \frac{1}{2}\ S(Sx, Sx, Tx)$$

$$S(STx, STx, TSx) \leq \phi(S(Sx, Sx, Tx)).$$

Lemma 2.7. [5]. Let ρt be a Lebesgue integrable function and ρ(t) > 0 for all t > 0. Let $F(x) = \int_0^x \psi(t)dt$, then F(x) is an increasing function in $[0, +\infty)$.

Definition2.8: [8]. Let S and T be two self-mappings on a set X. Any point x ∈ X is called coincidence point of S and T if Sx = Tx for some x ∈ X and we called u = Sx = Tx is a point of coincidence of S and T.

Definition 2.9 : A function $\emptyset : [0,\infty) \to [0,\infty)$ is called contractive modulus if it satisfy $\emptyset(t) \leq t$ for all $t \geq 0$.

Therorem:2.10: Let (X, S) be a complete S-metric space and P, T, f, g : X → X be self-mappings. If there exists an increasing function $Q : [0, \infty) \to [0, \infty)$ satisfying conditions for Altman also the following conditions holds:

(1) $P(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$;

(2) $$\int_0^{\alpha s(Px,Px,Ty)} P(t)dt \leq \phi\left(\int_0^{\phi(s(fx,fx,gy))} \phi(t)dt \right)$$

(3) $$\int_0^{\alpha s(fx,fx,Ty)]S(Px,Px,Ty) \leq \alpha\{S(Px,Px,Tx),S(Ty,Ty,gy),S(Px,Px,gy),S(Ty,Ty,fx)+f\{S(fx,fx,gy),S(Px,Px,fx),S(Ty,Ty,Ty),1/2(S(Ty,Ty,fx)+S(Px,Px,gy)\}} \phi(t)dt$$

holds for all x ∈ X.

If (P,f) and (T,g) are two pair S of continuous ϕ weakly commuting mappings then prove that P,T,f and g have a unique common fixed point in X:

3. Main results

We now state and prove our main theorem.

Theorem 3.1.

Let (X, S) be complete s – metric space and P, T,f,g : X→ X be self-mappings if there exists an increasing function Q:[0,∞) → [0, ∞) satisfying the following conditions from (1)-(3)

(3.1) P(X) ⊂ f(X), T(X) ⊂ g(X),

(3.2) $\int_0^{s(Px,px,Gy)} \psi(t)dt + \varphi \int_0^{S(Fx,Fx,Gy)S(Px,Px,Ty)} \psi(t)dt$

$\leq \varphi \int_0^{\max\{S(Fx,Fx,Gy)S(Px,Px,,Fx),S(Ty,Ty,Gy)^{(s(\quad Px,Px,Gy)+S(Ty,Ty,Fx),}}} \frac{}{2}(t)dt +$

$G\int_0^{\max\{S(Px,Px,Fx)S(Ty,Ty,Gy),S(Px,Px,Gy)(s(Ty,Ty,Fx)} \psi(t)dt$, for all x, y ∈ X, where φ is contractive modulus where (t) is a Lebesgue Integral function which is summable non negative and such that

(3.3.) $\int_0^{\delta} (t)$ dt $>$ o , $\delta >$ o

If (P ,F) and (T,G) are two pairs of continous ∅ - weekly commuting mappings. Then P,T,F,and G have a unique fixed point in X.

(i) (P,F) have a coincidence point.

(ii) (T, G) have a coincidence point.

(iii) Moreover, if both the pairs (A, F) and (T, G) are weakly compatible then P,T, F and G have aunique common fixed point.

Proof. choose $x_0 \in X$, then by (3.1) we can choose a sequence $\{x_n\}$ in X such that $x_0 = y_0$, $Px_{2n} = Gx_{2n+1} = y_{2n+1}$ and $Tx_{2n+1} = Fx_{2n+2} = y_{2n+2}$, for all n = 0, 1, 2, ...

We now show that the sequence $\{y_n\}$ defined above is a Cauchy sequence in X. Now we claim that Let us denote $d(y_n, y_{n+1})$ by S_n, for each n = 0, 1, 2, . . .

First, we show that $\int_0^{d_n} \psi(t)dt \le G(\int_0^{d_n} \psi(t)dt$. Now we claim that

$$\lim_{n\to\infty} S_n = 0$$

and then show that $\{y_n\}$ is a Cauchy sequence in X.

For this, putting x_{2n} for x and x_{2n+1} for y in (3.2),

we

$$\int_0^{(S_{2n+1}S_{2n+1},S_{2n+1})} \psi(t)dt + p\int_0^{(S_{2n}S_{2n}S_{2n+1})} \psi(t)dt \le p\int_0^{\max\{S_{2n}S_{2n}S_{2n+1},0\}} \psi(t)dt + G\left(\int_0^{\max\{(S_{2n},S_{2n,}S_{2n+1}),\frac{1}{2}d(y_{2n},y_{2n,}y_{2n+2})\}} \psi(t)dt\right)$$

i.e. $$\int_0^{(S_{2n+1},S_{2n+1})} \psi(t)dt \le G\left(\int_0^{\max\{S_{2n},S_{2n,}S_{2n+1},\frac{1}{2}d(y_{2n},y_{2n,}y_{2n+2})\}} \psi(t)dt\right)$$

But, from the triangle inequality for metric S, we have =

$$\frac{1}{2}s[y_{2n,}y_{2n}, y_{2n+2}] \le \frac{1}{2}[s(y_{2n,}y_{2n}, y_{2n+1}) + s(y_{2n+1,}y_{2n+1}, y_{2n+2})]$$

$$= \frac{1}{2}\left[s_{2n} + s_{2n} + s_{2n+1}\right] \leq \max\left(s_{2n}, s_{2n}, s_{2n+1}\right)$$

. Using this in above, we obtain

$$\int_0^{(s_{2n+1}s_{2n+1},S_{2n+1})} \psi(t)dt \leq \beta\left(\int_0^{\max(s_{2n},s_{2n},s_{2n+1})} \psi(t)dt\right)$$

$$= \int_0^{(S2n+1,2n+1)} \psi(t)dt \leq \beta(\max\{\int_0^{(S_{2n},2n)} \psi(t)dt \quad \int_0^{(S_{2n+1},2n+1)} \psi(t)dt\}$$

If we choose $\int_0^{s_{2n+1}} \psi(t)dt$ as "max" in above, then $s_{2n+1} > 0$ and we have

$$\int_0^{(s_{2n+1},S_{2n+1})} \psi(t)dt \leq \beta\left(\int_0^{(s_{2n+1},S_{2n+1})} \psi(t)dt\right) < \int_0^{s(s_{2n+1},S_{2n+1})} \psi(t)dt,$$

a contradiction. Hence,

$$\int_0^{(s_{2n+1},S_{2n+1})} \psi(t)dt \leq \beta \int_0^{(s_{2n},S_{2n})} \psi(t)dt, \tag{3.4}$$

Similarly, by setting x_{2n+2} for x and x_{2n+1} for y in(3.2), we obtain

$$\int_0^{(S_{2n}\,2S_{2n+2})} \psi(t)dt + \alpha \int_0^{(S_{2n}s_{2n+1}s_{2n+2})} \psi(t)dt \leq \alpha \int_0^{\max\{s_{2n}+2s_{2n+1},0\}} \psi(t)dt + \beta\left(\int_0^{\max\{s_{2n+1},s_{2n+2},\, s_{2n+1},\frac{1}{2}s(y_{2n+1},y_{2n+1}\, y_{2n+3})\}} \psi(t)dt\right)$$

i.e

$$\int_0^{(S_{2n+2},S_{2n+2},S_{2n+2})} \psi(t)dt \leq \phi\left(\int_0^{\max\{s_{2n+1},s_{2n+2},\, s_{2n+1},\frac{1}{2}s(y_{2n+1},y_{2n+1},\, y_{2n+3})\}} \psi(t)dt\right)$$

i.e $\int_0^{(s_{2n+2},S_{2n+2}s_{2n+2})} \psi(t)dt \le \beta\left(\int_0^{\max\{s_{2n+1},s_{2n+1}s_{2n+2},\}} \psi(t)dt\right)$

hence

(3.5) $$\int_0^{(s_{2n+2},S_{2n+2})} \psi(t)dt \le \beta\left(\int_0^{s_{2n+1}s_{2n+1}} \psi(t)dt\right)$$

Unifying *(*3.4) and (3.5), we obtain

$$\int_0^{ss_{n+1},S_{2n+1}} \psi(t)dt \le f\left(\int_0^{s_n s_n} \psi(t)dt\right) \quad \text{for all n = 0, 1, 2, ...}$$

Next, define a sequence $\{t_n\}$ by $t_{n+1} = f(t_n)$, with $t_1 = \int_0^{s_0} \psi(t)dt = \left(\int_0^{s(y_0,y_0,y_1)} \psi(t)dt\right)$

via assumption (a) that, $0 < f(t_n) = t < t_{n+1} < t_1$, $\forall n \ge 1$, if $t_1 > 0$. If $t_1 = 0$, then $t_n = 0$, for every n.

in addition, by induction, we show that $\int_0^{s_0} \psi(t)dt \le t_{n+1}$ for every n ∈ N

If n = 1, then by putting x_0 for x and x_1 for y in (3.2), we have

$$\int_0^{{}_1,y_1,y_2)} \psi(t)dt + \alpha \int_0^{s(y_0,y_1),s(y_1,y_1,y_2)} \psi(t)dt \le \alpha \int_0^{\max\{s(y_0,y_1)s(y_1,y_2),0\}} \psi(t)dt + f\left(\int_0^{\max\{s(y_0,y_1),s(y_1,y_2),\frac{1}{2}s(y_2,y_0)\}} \psi(t)dt\right)$$

Hence

$$\int_0^{s_{11}} \psi(t)dt = \left(\int_0^{s(y_1,y_1,y_2)} \psi(t)dt\right)$$

$$\leq \beta\left(\int_0^{\max\{s(y_0,y_0,y_1),s(y_1,y_{1,}y_2),\frac{1}{2}s(y_2,y_2,y_0)\}}\psi(t)dt\right)$$

$$= \beta\left(\int_0^{\max\{s(y_0,y_0,,y_1),s(y_1,y_1,y_2)\}}\psi(t)dt\right)$$

$$= \beta\left(\int_0^{s(y_0,y_0,y_1)}\psi(t)dt\right) = \beta\left(\int_0^{s_0}\psi(t)dt\right) = f(t_1) = t_2$$

because if we choose $s(y_1, y_1, y_2)$ as "max " then $s(y_1, y_1, y_2) > 0$ and it yields

$\int_0^{s_1}\psi(t)dt \leq \beta(\int_0^{s_1}\psi(t)dt) < \int_0^{s_1}\psi(t)dt$ which is a contradiction.

Thus, for n = 1, we observe that

$$\int_0^{s_1}\psi(t)dt \leq t_2$$

Assume, for some fixed n, that

$\int_0^{s_n}\psi(t)dt \leq t_{n+1}$ is true.

subsequently, by induction; we have, since β is non decreasing,

$$\int_0^{s_{n+1}}\psi(t)dt \leq \beta\left(\int_0^{s_n}\psi(t)dt\right) \leq \beta(t_{n+1}) = t_{n+2}.$$

Thus, it follows that

$$\int_0^{s_{n1}} \psi(t)dt \le t_{n+1} \quad \text{for all } n \in N.$$

if $t_1 = 0$, then $s_n = 0$ for every n, so that we consider the case where $t_n > 0$, for every n.

Now, by conditions (a)–(c) and $t_{n+1} = (t_n)$, $n \in N$, which shows that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = 0$, it follows that $\{y_n\}$ is a Cauchy sequence. certainly, if m, n ∈ N with $m \ge n$, then using that ψ is a nonincreasing implies

$$\int_0^{\sum_{k=n}^{m-1} s_k} \psi(t)dt = \int_0^{d_n} \psi(t)dt + \int_{s_n}^{s_n+s_{n+1}} \psi(t)dt + \int_{s_n+s_{n+1}}^{s_n+s_{n+1}+s_{n+2}} \psi(t)dt + \dots\dots\dots + \int_{\sum_{k=n}^{m-2} s_k}^{\sum_{k=n}^{m-1} s_k} \psi(t)dt$$

$$\le \int_0^{s_n} \psi(t)dt + \int_0^{s_{n+1}} \psi(t)dt + \int_0^{s_{n+2}} \psi(t)dt + \dots\dots\dots + \int_0^{s_{m-1}} \psi(t)dt$$

$$= \sum_{k=n}^{m-1}\left(\int_0^{s_k} \psi(t)dt\right),$$

We obtain

$$\int_0^{s(y_m, y_n)} \psi(t)dt \le \left(\int_0^{\sum_{k=n}^{m-1} s_k} \psi(t)dt\right) \le \sum_{k=n}^{m-1}\left(\int_0^{s_k} \psi(t)dt\right) \le \sum_{k=n}^{m-1}(t_{k+1})$$

$$= \sum_{k=n+1}^{m}(t_k) = \sum_{k=n+1}^{m}(t_k)\frac{t_k(t_k - t_{k+1})}{t_k - f(t_k)} \le \sum_{k=n+1}^{m}\left(\int_{t_{k+1}}^{t_k} G(t)dt\right) \le \left(\int_{t_{m+1}}^{t_{n+1}} G(t)dt\right).$$

Since the sequence $\{t_n\}$ is convergent and

$\left(\int_0^T G(t)dt\right) < +\infty$ for each $\tau \in \left(0, \int_0^K \psi(t)dt\right)$ where $r \subseteq [0,k]$,then the last term tends to zero as $n \to \infty$ and hence $\{y_n\}$ } is a Cauchy sequence in X.

Now we suppose that the range of one of the mappings is complete.

Case 1: Suppose that G(X) is a complete sub space of X, then the sub sequence{ $y_{2n+1}\}=\{G\, x_{2n+1}\}$ is cauchy sequence in G(X) and hence converges to a limit, say z in X. Since $\{y_n\}$ } is a Cauchy and its sub sequence $\{y_{2n+1}\}$ is convergent to z, so $\{y_n\}$ is also converges to z. Hence its sub sequence $\{y_{2n+2}\}$ is also convergent to z. Thus we have

$$\lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} Fx_{2n} = z .$$

Let $v \in F^{-1}z$, then gv=z .We claim that Tv =z for this, setting x = x_{2n} and y = v in the implicit relation (3.2) we have

$$\int_0^{s(Px_{2n},Px_{2n},Tv)} \psi(t)dt + \alpha \int_0^{s(fx_{2n},fx_{2n},gv)s(Px_{2n},,Px_{2n},Tv)} \psi(t)dt$$

$$\leq$$

$$\alpha \int_0^{\max\{s(Px_{2n},Px_{2n},fx_{2n})s(Tv,gv),s(Px_{2n},Px_{2n},gv)s(Tv,Tv,fx_{2n})\}} \psi(t)dt + \beta\left(\int_0^{\max\{(fx_{2n},fx_{2n},gv),s(Px_{2n},Px_{2n},fx_{2n}),s(Tv,Tv,gv),\frac{1}{2}(s(Px_{2n},Px_{2n},gv)\}+s(Tv,Tv,fx_{2n}))\}} \psi(t)dt\right)$$

If we suppose that s(z, Tv) > 0, then we have, for n large enough,

$$\int_0^{s(,,Px_{2n},Tv)} \psi(t)dt + \alpha \int_0^{s(fx_{2n},,fx_{2n},gv)s(Px_{2n},,Px_{2n},Tv)} \psi(t)dt \leq$$

$$\alpha \int_0^{\max\{s(Px_{2n},Px_{2n},fx_{2n})s(Tv,Tv,gv),s(Px_{2n},Px_{2n},gv)s(Tv,Tv,fx_{2n})\}} \psi(t)dt + \beta\left(\int_0^{s(Tv,Tv,z)} \psi(t)dt\right)$$

Letting n $\to\infty$, it yields

$$\int_0^{s(z,Tv)} \psi(t)dt \le f\left(\int_0^{s(z,Tv)} \psi(t)dt\right) < \left(\int_0^{s(z,Tv)} \psi(t)dt\right)$$

which is a contradiction. Thus s(Tv, z) = 0, so that Tv = z. Hence z = Tv = Gv, show that v is a coincidence point of T and G.

Further, since T(X) ⊂ g(X), Tv = z implies that z ∈ f(X).

Let u ∈ F^{-1} z, then Fu = z. Now, we claim that Pu = z. For this, putting x = u and y = v in (3.2), we have

$$\int_0^{s(Au,Au,z)} \psi(t)dt + p\int_0^{0,s(Pu,Pu,z)} \psi(t)dt \le \alpha\left(\int_0^{\max(0,0,0)} \psi(t)dt\right) + \beta\left(\int_0^{\max\{0,s(Pu,Pu,z,0.\frac{1}{2}s(Pu,z)\}} \psi(t)dt\right)$$

i.e., $\int_0^{s(Pu,,Pu,z)} \psi(t)dt \le \beta\int_0^{s(Pu,Pu,z)} \psi(t)dt < \int_0^{s(Pu,Pu,,z)} \psi(t)dt$,

if s(Pu,z)> 0 getting a contradiction. Thus Pu = z. Hence z = Pu = Su, showing that u is a coincidence point of (P,f).

Case II. If we assume S(X) to be a complete subspace of X, then analogous arguments establish the earlier conclusion. Indeed, in this case, the subsequence $\{y_{2n+2}\} = \{Fx_{2n+2}\}$ is a Cauchy sequence in F(X) and hence converges to a limit, say z in β(X). Similarly to Case I,

$\lim_{n\to\infty} Gx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} px_{2n} = \lim_{n\to\infty} \beta x_{2n} = z$. Let v ∈ X be such that βv = z. To prove that

pv = z, we take x = v and y = x_{2n+1} in the implicit relation (3.2), hence, assuming that s(pv, z) > 0, we get, for n large enough,

$$\int_0^{s(Pv,Pv,Tx_{2n+1})} \psi(t)dt + p \int_0^{s(fv,fv,gx_{2n+1})s(Pv,Pv,Tx_{2n+1},)} \psi(t)dt \leq$$

$$\alpha \int_0^{\max\{s(Pv,Pv,z),s(Tx_{2n+1},Tx_{2n+1},gx_{2n+1}),s(Pv,Pv,gx_{2n+1}),s(Tx_{2n+1},Tx_{2n+1},fv)\}} \psi(t)dt + f\left(\int_0^{s(Pv,Pv,z\}} \psi(t)dt\right)$$

hence, taking the limit as n $\rightarrow \infty$, we obtain

$$\int_0^{s(Pv,Pv,z)} \psi(t)dt \leq f \int_0^{s(Pv,Pv,z)} \psi(t)dt < \int_0^{s(Pv,Pv,z)} \psi(t)dt$$

which is a contradiction. Hence Pv = βv = z.

On the other hand, since P(X) ⊂ G(X), then z = gu, for some u ∈ X. To check that Tu = z, we take x = v and y = u in (3.2), achieving

$$\int_0^{s(z,z,u)} \psi(t)dt \leq f \int_0^{s(Tv,Tv,z)} \psi(t)dt < \int_0^{s(Tv,Tv,z)} \psi(t)dt$$

If s(Tu,z)>0, getting a contradiction. This proves that Tu = Gu = z.

The remaining two cases are essentially the same as the previous cases. Indeed, if P(X) is complete, then by (3.1), z ∈P(X) ⊂ G(X). Similarly, if T(X) is complete, then z ∈ T(X) ⊂F(X).Thus pairs (P,F) and (T,G) have coincidence points. Hence in all we have z = Pu = Fu = Tv =v. This proves our assertions in (i) and (ii). Now, the weak compatibility of (P,F) gives Pz = P

Fu = FPu = Fz; i.e., Pz = Sz. Similarly, the weak compatibility of (T,G) gives Tz = TGv = GTv = Gz; i.e., Tz = Gz.

To show that z is a coincidence point of P,T,F and G, we have to check that Pz = Tz.

For this, putting x = z and y = z in (4.2), we have

$$\int_0^{s(Pz,Pz,Tz)} \psi(t)dt + \alpha \int_0^{s(fz,fz,z)s(Pz,Pz,Tz)} \psi(t)dt \le$$

$$\alpha \int_0^{\max\{s(Pz,Pz,z),s(Tz,Tz,gz),s(Pz,Pz,gz),s(Tz,Tz,fz)\}} \psi(t)dt + \beta \left(\int_0^{\max\{s(fz,fz,gz),s(Pz,Pz,fz),s(Tz,Tz,gz),\frac{1}{2}[s(Pz,Pz,gz)+s(Tz,Tz,fz)]\}} \psi(t)dt \right)$$

i.e $\int_0^{s(Pz,Pz,Tz)} \psi(t)dt \le \beta \int_0^{s(Pz,Pz,Tz)} \psi(t)dt < \int_0^{s(Pz,Pz,Tz)} \psi(t)dt$ if s(Pz,Pz,Tz)>0, which is a contradiction.

Thus Pz = Tz. Hence Pz = Fz = Tz = Gz.

To show that z is a common fixed point, putting x = z and y = v in (4.2), we have

$$\int_0^{s(Pz,Pz,Tv)} \psi(t)dt + \alpha \int_0^{s(fz,fz,gv)s(Pz,Pz,Tv)} \psi(t)dt \le$$

$$\alpha \int_0^{\max\{s(Pz,Pz,z),s(Tv,Tv,gv),s(Pz,Pz,gv),s(Tv,Tv,fz)\}} \psi(t)dt + f \left(\int_0^{\max\{s(f,fz,gv),s(Pz,Pz,fz),s(Tv,Tv,gv),\frac{1}{2}[s(Pz,Pz,gv)+s(Tv,Tv,fz)]\}} \psi(t)dt \right)$$

i.e $\int_0^{s(Pz,Pz,z)} \psi(t)dt \le f \int_0^{s(Pz.Pz,z)} \psi(t)dt < \int_0^{s(Pz,Pz,z)} \psi(t)dt$

if d(Pz, z) > 0, getting a contradiction.

Thus, we obtain z = Pz = Tz = Fz = Gz. Uniqueness of common fixed point z follows easily by (3.2). This completes the proof. We remark that F in Theorem 3.1 must be defined, at least, in $\left[0, \int_0^K \psi(s)ds\right]$ where cl(ran d) $\subset$ [0, K].

If we take $\psi : R_+ \to R_+$ condition ψ is a nonincreasing function then ψ is measurable, summable on each compact interval, and condition (4.3) holds if $\int_0^{\in} \psi(t)dt$ is positive and finite for an $\in>0$.

Note that condition ψ is a nonincreasing function is valid for constant functions ψ, but it is not true for functions of the type $\psi(t) = Rt$, $t > 0$, where $R > 0$.

Theorem 3.2. In Theorem 3.1, condition ψ is a nonincreasing function can be replaced by the following one:

$\psi(t) > 0$, $\forall t > 0$, and $f\int_0^{x} \psi(t)dt \le \int_0^{f(x)} \psi(t)dt$, $\forall x > 0$.

Proof. We have to justify that the sequence $\{y_n\}$ defined in the proof of Theorem 3.1 is a Cauchy sequence. Using that $\int_0^{s_{n+1}} \psi(t)dt \le f \int_0^{d_n} \psi(t)dt$, for all n = 0, 1, 2, . . .,

and $\psi(t) > 0$, $\forall t > 0$, and $f\int_0^{x} \psi(t)dt \le \int_0^{f(x)} \psi(t)dt$, $\forall x > 0$.we get

$\int_0^{s_{n+1}} \psi(t)dt \le \int_0^{f(s_n)} \psi(t)dt$, for all n = 0, 1, 2, . . ., and $s_{n+1} \le f(d_n)$, for all n = 0, 1, 2, ...

We define a sequence $\{t_n\}$ by $t_1 = s_0$, $t_{n+1} = f(t_n)$, $\forall n \in N$. If $t_1 = d_0 = 0$, then $s_n = 0$ for every n. Consider $t_1 > 0$, hence $t_{n+1} = f(t_n) < t_n$, $\forall n \in N$ and $t_n \to 0$. Besides, it can be easily obtained that $s_n \le t_{n+1}$, for all n = 0, 1, 2, ... Now, for m, n $\in$ N with $m \ge n$, we get

$s(y_m, y_n) \le \sum_{k=n}^{k=m-1} s_k \le \sum_{k=n}^{k=m-1} t_{k+1} = \sum_{k=n+1}^{k=m} t_k \le \int_{t_{m+1}}^{t_{n+1}} g(t)dt$ and the sequence $\{y_n\}$ is a Cauchy sequence, since $\int_0^{\tau} g(t)dt < +\infty$

for each $\tau > 0$. Note that condition $f\int_0^{x} \psi(t)dt \leq \int_0^{f(x)} \psi(t)dt$, $\forall x > 0$, is trivially satisfied if $\psi \equiv 1$ and reduces to $f(Rx) \leq RF(x)$, $\forall x > 0$, if $\psi \equiv R$.

In fact such condition can be dropped, as established in the following result.

Theorem 3.3. In Theorem 3.1, ψ is a nonincreasing function can be replaced by the following one: $\psi(t) > 0$, for every $t > 0$.

Corollary 3.4. Let P,T,f, and g be four self-mappings of a metric space (X, s) satisfying (3.1) and (3.6)

$$\int_0^{s(Px,Px,Ty)} \psi(t)dt \leq G\left(\int_0^{\max\{ s(fx,fx,gv), s(Px,,Px,x), s(Ty,Ty,gy), \frac{1}{2}[s(Px,Px,gy)+s(Ty,Ty,fx)]\}} \psi(t)dt \right)$$

for all x, y ∈ X, where $f : [0, +\infty) \to R$ is non decreasing and satisfies the Altman type conditions (a)–(c) and $\psi : R_+ \to R_+$ is a non negative, Lebesgue measurable mapping which is summable on each compact interval, and satisfies (3.3). Assume that one of the hypotheses , ψ is a nonincreasing function ,$\psi(t) > 0$, $\forall t > 0$, and $f\int_0^{x} \psi(t)dt \leq \int_0^{f(x)} \psi(t)dt$, $\forall x > 0$.

or $\psi(t) > 0$, for every $t > 0$ holds. If one of P(X), T(X), F(X) or G(X) is a complete subspace of X, then

(i) (P,F) have a coincidence point.

(ii) (T,G) have a coincidence point.

Moreover, if both the pairs (P,G) and (T,G) are weakly compatible then P,T,F and G have a unique common fixed point.

Corollary 3.5. Let $\{P_i\}$ i∈N,F and G be self-mappings of a metric space (X, s) such that (3.7)

$P_i(X)\} \subset G(X)$, $P_{i+1}(X) \subset F(X)$ (3.8)

$$\int_0^{s(,s_i x, s_i x, s_{i+1} y)} \psi(t)dt + \alpha \int_0^{s(fx, fx, gy) s(s_i x, s_i x, s_{i+1} y)} \psi(t)dt \leq$$

$$\alpha \int_0^{\max\{s(P_i x, P_i x, fx), s(P_{i+1} y, P_{i+1} y, y), s(P_i x, P_i x, gy), s((P_{i+1} y, P_{i+1} y, fx)\}} \psi(t)dt + f\left(\int_0^{\max\{s(fx, fx, gy), s(Px, Px, fx), s(P_{i+1} y, P_{i+1} y, gy)), \frac{1}{2}[s(P_i x, P_i x, gy) + s(Py, Py, fx)]\}} \psi(t)dt \right)$$

for all x, y ∈ X, where $\alpha \geq 0$, $f : [0, +\infty) \to R$ is non decreasing and satisfies the Altman's conditions (a)–(c) and $\psi : R_+ \to R_+$ is a non negative, Lebesgue measurable mapping which is summable on each compact interval, and such that (3.3) holds. Assume that one of the ψ is a nonincreasing function $\psi(t) > 0$, $\forall t > 0$,

and $f \int_0^x \psi(t)dt \leq \int_0^{f(x)} \psi(t)dt$, $\forall x > 0$.

or $\psi(t) > 0$, for every $t > 0$holds. If one of P_i (X), F(X) or G(X) is a complete subspace of X, and if the pairs (P_i , F) and ({P_{i+1}, G) are weakly compatible, then $\{P_i\}$ i∈N, f and g have a unique common fixed point.

.**Corollary 3.6**. Let f and g be self-maps of a metric space (X, s). Let $\{P_i\}$i∈N and $\{T_i\}$i∈N be two sequences of self-mappings of the metric space (X, s) satisfying the conditions: (3.9) $P_i(X) \subset G(X)$, $T_i(X) \subset F(X)$, (3.10)

$$\int_0^{s(P_ix,P_ix,T_iy)} \psi(t)dt + \alpha \int_0^{s(fx,fx,gy)s(P_ix,P_ix,T_iy)} \psi(t)dt \le$$

$$\alpha \int_0^{\max\{s(P_ix,P_ix,fx),s(T_iy,T_iy,gy),s(Px,Px,gy),s(T_iy,T_iy,fx)\}} \psi(t)dt + f\left(\int_0^{m(x,x,y)} \psi(t)dt\right)$$

for all x, y ∈ X, where $\alpha \ge 0$, $f : [0, +\infty) \to R$ is non decreasing and satisfies the Altman type conditions (a)–(c), $\psi : R_+ \to R_+$ is a non negative, Lebesgue measurable mapping which is summable on each compact interval, and such that (3.3) holds, and $m(x, y) = \max\{s(fx,fx, Gy), s(P_ix, P_ix, Fx), s(T_iy, T_iy, Gy), \frac{1}{2}[s(P_ix, P_ix, Gy) + s(T_iy, T_iy, Fx)]\}$.

Assume that one of the ψ is a nonincreasing function $\psi(t) > 0$, $\forall t > 0$, and F

$$\int_0^{x} \psi(t)dt \le \int_0^{f(x)} \psi(t)dt, \ \forall x > 0.$$

or $\psi(t) > 0$, for every $t > 0$holds. If one of$P_i(X)$, $T_i(X)$, F(X) or G(X) is a complete subspace of X, then

(i)(P_i ,F) have a coincidence point

(ii) (T_i ,G) have a coincidence point.

Moreover, if both the pairs (P_i,f) and (T_i ,G) are weakly compatible thenP_i, T_i, f and g have a unique common fixed point. Now we give an example to show the validity of the main results Theorems 3.1–3.3.

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