# Some Properties of the Concircular Curvature Tensor on LP-Sasakian Manifolds

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**Abstract:**The present paper is an attempt to study the geometrical properties of the concircular curvature tensor on LP-Sasakian manifolds with generalised Tanaka-Webster connection.In this manner, we have shown that if an n- dimensional LP-Sasakian manifold is $\xi$-concircularly flat with respect to generalized Tanaka-Webster connection, then its scalar curvature with respect to generalized Tanaka-Webster connection vanishes. Also, we have proved that if an $n$- dimensional LP-Sasakian manifold satisfies the condition, $\tilde{C}(X,Y).\tilde{S}=0$, with respect to generalized Tanaka-Webster connection, then the scalar curvature with respect to generalized Tanaka-Webster connection vanishes.

**Keywords:**Concircular curvature tensor, Einstein manifold, LP-Sasakian manifold, generalized Tanaka-Webster connection.

**1.Introduction**

A transformation of a Riemannian manifold $M^n$of dimension $n$, which transforms every geodesic circle of $M^n$ into a geadesic circle, is called a concircular transformation [11]. Here, geodesic circle means a curve in $M^n$ whose first curvature is constant and whose second curvature is identically zero. An interesting invariant of a concircular transformation is the concircular curvature tensor $C$. It is defined by [12]

$$C(X,Y)Z=K(X,Y)Z-\frac{r}{n(n-1)}\{g(Y,Z)X-g(X,Z)Y\}. \qquad (1.1)$$

Here, $K$ is the curvature tensor with Riemannian connection $D, r$is the scalar curvature and $X,Y,Z$ are vector fields on $M^n$.

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Therefore, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. Many geometers studied and investigated various properties of concircular curvature tensor in different manifolds. Recently, Pandey et al. [7] studied and found several properties of concircular curvature tensor on LP-Sasakian manifold endowed with the quarter-symmetric non-metric connection and they provided examples which satisfied the condition of $\xi$-concircularly flat and $\phi$-concircularly flat LP-Sasakian manifold equipped with quarter-symmetric non-metric connection. Thereafter, Kiran et al. [4] studied concircular curvature tensor of Kenmotsu manifold with respect to generalized Tanaka-Webster connection. Meanwhile, De et al. [2] were investigating the effects of concircular flatness and concircular symmetry of a warped product manifold on its fibre and base manifolds. Most recently, Chavan [1] discussed and found many results on concircular curvature tensor of generalized Sasakian- space-forms and various examples were constructed to verify some results.

On the other hand, Matsumoto [5] introduced the idea of Lorentzian Para-Sasakianmanifolds, an almost para-contact manifold was introduced by [9]. Later, Mihai and Rosca [6] introduced the same notion independently and obtained various results. Motivated by these studies and many authors, the main purpose of this paper is to study concircular curvature tensor on LP-Sasakian manifold with generalized Tanaka-Webster connection.

The article is presented as follows. After introduction, it is devoted to the preliminaries. In section 3, we discuss some results of curvature tensor on LP-Sasakian manifold with generalized Tanaka-Webster connection. Finally, we characterize the concircular curvature tensor with respect to generalized Tanaka-Webster connection on LP-Sasakian manifold.

**2. Preliminaries**

A differentiable manifold $M^n$of n-dimension, is termed as an almost para-contact manifold, if it admits an almost para-contact structure $(\phi,\ \xi,\ \eta,\ g)$consisting of a (1,1) tensor field $\phi$, vector field $\xi$, 1-form $\eta$and Lorentzian metric $g$ satisfying

$$\phi o\xi = 0, \quad \eta o\phi = 0, \quad \eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(D_X\eta)(Y) = g(X,\phi Y) = (D_Y\eta)X, \tag{2.4}$$

for any vector fields $X,Y$on $M^n$. Such a manifold is called as Lorentzian para-contact manifold and the structure $(\phi,\ \xi,\ \eta,\ g)$a Lorentzian para-contact structure [5].

Addition to the above, if $(\phi,\ \xi,\ \eta,\ g)$ satisfies

$$d\eta = 0, \quad D_X\xi = \phi X, \tag{2.5}$$

$$(D_X\phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

for $X, Y$ tangent vectors to $M^n$ , then $M^n$ is called a Lorentzian para-Sasakian(shortly LP-Sasakian) manifold, where $D$ is the covariant differentiation with respect to Lorentzian metric $g$.
Moreover, in a LP-Sasakian manifold $M^n$ the following relations hold [8]

$$\eta(K(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.7}$$

$$K(\xi,X)Y - g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$K(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.10}$$

for all vector fields $X, Y, Z$ on the manifold $M^n$.

## 3. Curvature tensor of LP-Sasakian manifold with respect to generalized Tanaka- Webster connection

The generalized Tanaka-Webster connection $\widetilde{D}$ defined by Tano [10] for contact metric manifolds is given as ,

$$\widetilde{D}_X Y = D_X Y + (D_X \eta)(Y)\xi - \eta(Y) D_X \xi + \eta(X)\phi Y, \tag{3.1}$$

for any $X, Y$ tangent to $M^n$.
By virtue of (2.4) and (2.5), the above equation can be written as

$$\widetilde{D}_X Y = D_X Y + g(X, \phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{3.2}$$

The curvature tensor $\widetilde{K}$ with respect to generalized Tanaka-Webster connection $\widetilde{D}$ is given as [3]

$$\widetilde{K}(X,Y)Z = K(X,Y)Z + 3g(Y,\phi Z)\phi X - 3g(X,\phi Z)\phi Y + 3\eta(Y)g(X,Z)\xi$$
$$-3\eta(X)g(Y,Z)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \tag{3.3}$$

where $K$ is the curvature tensor with respect to Riemannian connection $D$.

One can easily find the following relations

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-1)\eta(Y)\eta(Z) + 3ag(Y,\phi Z), \tag{3.4}$$

$$\tilde{r} = r - n + 1 + 3a^2, \tag{3.5}$$

$$\tilde{Q}Y = QY + (n-1)\eta(Y)\xi + 3a\phi Y \tag{3.6}$$

where $a = trace\ \phi$.

**Proposition3.1**:If a LP-Sasakian manifold $M^n$ admits generalized Tanaka-Webster connection whose curvature tensor vanishes, then the scalar curvature $r$ with respect to Riemannian connection is given as

$$r = n - 1 - 3a^2.$$

**Theorem 3.1**:In a LP-Sasakian manifold $M^n$ with generalized Tanaka-Webster connection if the relation (3.7) holds, then the manifold is an Einstein manifold for the connection $\widetilde{D}$if and only if it is an Einstein manifold for the connection $D$.

Proof:

A LP-Sasakian manifold $M^n$ is said to be an Einstein manifold with respect to Riemannian connection if

$S(Y,Z) = \frac{r}{n} g(Y,Z).$

Analogous to this definition, we define Einstein manifold with respect to generalized Tanaka-Webster connection by

$$\tilde{S}(Y,Z) = \frac{\tilde{r}}{n} g(Y,Z). \tag{3.7}$$

From (3.4), (3.5) and (3.7),

$$\tilde{S}(Y,Z) - \frac{\tilde{r}}{n} g(Y,Z) = S(Y,Z) - \frac{r}{n} g(Y,Z) - (n-1)\eta(Y)\eta(Z) + 3ag(Y,\phi Z) + \frac{(n-1-3a^2)}{n} g(Y,Z). \tag{3.8}$$

If

$$-(n-1)\eta(Y)\eta(Z) + 3ag(Y,\phi\, Z) + \frac{(n-1-3a^2)}{n} g(Y,Z) = 0,$$

which implies that

$$3ag(Y,\phi Z) = (n-1)\eta(Y)\eta(Z) - \frac{(n-1-3a^2)}{n} g(Y,Z). \tag{3.9}$$

In consequence of the equation (3.9) and (3.8), we get

$\tilde{S}(Y,Z) - \frac{\tilde{r}}{n} g(Y,Z) = S(Y,Z) - \frac{r}{n} g(Y,Z).$

This gives the theorem.

**4.Concircular curvature tensor with respect to generalized Tanaka-Webster connection on LP-Sasakian manifold**

Analogous to the definition, Concircular curvature tensor with respect to generalized Tanaka-Webster connection on LP-Sasakian manifold is given as

$$\tilde{C}(X,Y)Z = \tilde{K}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \qquad (4.1)$$

for all vector fields $X, Y, Z$, where $\tilde{K}$ is curvature tensor and $\tilde{r}$ is scalar curvature with respect to generalized Tanaka-Webster connection respectively.

**Theorem 4.1**:In a LP-Sasakian manifold, concircular curvature tensor with respect to generalized Tanaka-Webster connection $\tilde{D}$ has the following properties

(i) $\tilde{C}(X,Y)Z) + \tilde{C}(Y,X)Z = 0$,

(ii) $\tilde{C}(X,Y)Z + \tilde{C}(Y,Z)X + \tilde{C}(Z,X)Y = 0$,

(iii) $'\tilde{C}(X,Y,Z,U) - {}'\tilde{C}(Z,U,X,Y) = 0$, if $X, Y, Z, U$ are orthogonal to $\xi$, (4.2)

where$'\tilde{C}(X,Y,Z,U) = g(\tilde{C}(X,Y)Z,U)$. (4.3)

Proof: By interchanging $X$ and $Y$ in (4.1) and adding to the (4.1) immediately gives (4.2)(i).

By using (3.3), (4.1) and first Bianchi identity

$$K(X,Y)Z + K(Y,Z)X + K(Z,X)Y = 0 \qquad (4.4)$$

with respect to Riemannian connection $D$, immediately we get (4.2)(ii).

By virtue of (4.1), (3.3), (4.3), the relations $\widetilde{'K}(X,Y,Z,U) = g(\tilde{K}(X,Y)Z,U)$and $\widetilde{'K}(X,Y,Z,U) = \widetilde{'K}(Z,U,X,Y)$, we have

$$'\tilde{C}(X,Y,Z,U) - {}'\tilde{C}(Z,U,X,Y) = 4\eta(Y)\eta(Z)g(X,U) - 4\eta(X)\eta(U)g(Y,Z).$$

Suppose $X, Y, Z, U$ are orthogonal to $\xi$. Then the above equation implies the equation (4.2)(iii).

**Theorem 4.2**: An n-dimensional LP-Sasakian manifold is $\xi$-concircularly flat with respect to the generalized Tanaka-Webster connection if and only if the manifold is $\xi$-concircularly flat with respect to Riemannian connection provided the vector fields $X,\ Y$ orthogonal to $\xi$.

Proof:Putting $Z = \xi$ in (4.1) and using (3.3), (2.1) and (3.5), we get

$$\tilde{C}(X,Y)\xi = C(X,Y)\xi - \frac{(n^2 - 2n + 1 + 3a^2)}{n(n-1)}\{\eta(Y)X - \eta(X)Y\}.$$

$= C(X,Y)\xi$,if$X, Y$ are orthogonal to $\xi$. (4.5)

This gives the theorem.

**Theorem 4.3:** If a n-dimensional LP-Sasakian manifold $M^n$ is $\xi$-concircularly flat with respect to generalized Tanaka-Webster connection, then the scalar curvature with respect to generalized Tanaka-Webster connection vanishes..

Proof: Setting $Z = \xi$ in (4.1), we get

$$\tilde{C}(X,Y)\xi = \tilde{K}(X,Y)\xi - \frac{\tilde{r}}{n(n-1)}\{\eta(Y)X - \eta(X)Y\}. \quad (4.6)$$

If $M^n$ is $\xi$-concircularly flat with respect to generalized Tanaka-Webster connection, then it follows from (4.6) that

$$\tilde{K}(X,Y)\xi = \frac{\tilde{r}}{n(n-1)}\{\eta(Y)X - \eta(X)Y\}. \quad (4.7)$$

In consequence of (3.3), (2.9) and (4.7), we obtain

$$\frac{\tilde{r}}{n(n-1)}\{\eta(Y)X - \eta(X)Y\} = 0$$

which implies that

$\tilde{r} = 0.$

Thus we get the theorem.

**Theorem4.4**: If an n-dimensional LP-Sasakian manifold satisfies the condition,

$\tilde{C}(X,Y).\tilde{S} = 0$ with respect to generalized Tanaka-Webster connection, then the scalar curvature with respect to generalized Tanaka-Webster connection vanishes.

Proof: Now, suppose $\tilde{C}(X,Y).\tilde{S} = 0.$

Then we have

$$\tilde{S}(\tilde{C}(X,Y)U,V) + \tilde{S}(U,\tilde{C}(X,Y)V) = 0, \quad (4.8)$$

forall $X,Y,U,V \in \chi(M^n)$.

Replacing $X$ by $\xi$ in the above equation, we get

$$\tilde{S}(\tilde{C}(\xi,Y)U,V) + \tilde{S}(U,\tilde{C}(\xi,Y)V) = 0. \quad (4.9)$$

In view of (2.1), (4.1) and (3.3), the above equation becomes

$$\frac{\tilde{r}}{n(n-1)}\{\eta(U)\tilde{S}(Y,V) - \eta(V)\tilde{S}(U,Y)\} = 0 \quad (4.10)$$

which implies that $\tilde{r} = 0$.

Hence, we complete the proof.

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