Some Curvature Conditions on LP-Sasakian Manifolds

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Abstract: The present paper deals with LP-Sasakian manifolds equipped with generalized Tanaka-Webster connection. Here, we have shown that the m-projective curvature tensor and concircular curvature tensor of LP-Sasakian manifold with respect to generalized Tanaka-Webster connection,provided the vector fields are orthogonal to ξ,are linearly dependent if and only if the manifold is an η-Einstein manifold.Later, we have proved that if M_n is n-dimensional ϕ-concircularly flat LP-Sasakian manifold with respect to generalized Tanaka-Webster connection, provided the vector fields are orthogonal to ξ, then M_n is an η-Einstien manifold.

Keywords: Curvature tensors, Einstein manifold, LP-Sasakian manifold, generalized Tanaka-Webster connection.

1. Introduction

The Tanaka-Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-Manifold[3,8]. Tanno[9] defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. Also, a cononicalparacontact connection on a paracontact metric manifold which seems to be the paracontact analogue of the (generalized) Tanaka-Webster connection had been defined by Zamkovoy in [13]. Later, Ghosh and De [2] studied the Tanaka-Webster connection associated to a Kenmotsu structure with the help of g-Tanaka-Webster connection and they found various curvature properties on Kenmotsu manifolds.Recently, Kazan and Karadagi [3] studied the curvature tensors of a trans-Sasakian manifold with the generalized Tanaka-Webster connection and investigated some special curvature conditions of a trans-Sasakian manifold with connections. Thereafter, $\ddot{U}$nal and Altin [10] characterized N(K)-contact metric manifolds with the generalized Tanaka-Webster connection and proved that if an N(K)-contact metric manifold admitting this connection was K-contact, then it was anexample of the generalized Sasakian space form.

On another hand, the idea of Lorentzian Para-Sasakian manifold was introduced by Matsumato [4]. Later,Mihai and Rasca [5] also defined the same idea independently and they investigated various results on this manifold. Many researchers studied on LP-Sasakian manifold with several connections and investigated variousproperties. For instances, Devi et al. [1] found certain curvature properties on Lorentzian Para-Sasakian manifolds equipped with generalized Tanaka-Webster connection.

The generalized Tanaka-Webster connection$\widetilde{\nabla}$[3],for a contact metric manifold is defined as
$$\widetilde{\nabla}_Y Z = \nabla_Y Z + (\nabla_Y \eta)(Z)\xi - \eta(Z)\nabla_Y \xi + \eta(Y)\phi Z, \quad (1.1)$$

for all $Y, Z \in \chi(M_n)$.

In a LP-Sasakian manifold M_n of dimension $(n > 2)$, the projective curvature tensor$P$ [6], m-projective curvature tensor W^*[7], concircular curvature tensor C[12] with respect to Riemannian connection ∇, are given by

$$P(Y,Z)\,U = K(Y,Z)U - \frac{1}{n-1}\{S(Z,U)Y - S(Y,U)Z\}, \quad (1.2)$$

$$W^*(Y,Z)\,U = K(Y,Z)U - \frac{1}{2(n-1)}\{S(Z,U)Y - S(Y,U)Z + g(Z,U)QY - g(Y,U)QZ\}, \quad (1.3)$$

$$C(Y,Z)\,U = K(Y,Z)U - \frac{r}{n(n-1)}\{g(Z,U)Y - g(Y,U)Z\}, \quad (1.4)$$

for all $Y, Z, U \in \chi\,(M_n)$,where K, S, Q, r are Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature with respect to Riemannian connection.

Analogous to the definition given above, the projective curvature tensor $\tilde{P}$, m-projective curvature tensor $\widetilde{W^*}$ and concircular curvature $\tilde{C}$ with respect to generalized Tanaka-Webster connection$\widetilde{\nabla}$, are given by

$$\tilde{P}(Y,Z)\,U = \widetilde{K}(Y,Z)U - \frac{1}{n-1}\{\tilde{S}(Z,U)Y - \tilde{S}(Y,U)Z\}, \quad (1.5)$$

$$\widetilde{W^*}(Y,Z)\,U = \widetilde{K}(Y,Z)U - \frac{1}{2(n-1)}\{\tilde{S}(Z,U)Y - \tilde{S}(Y,U)Z + g(Z,U)\tilde{Q}Y - g(Y,U)\tilde{Q}Z\}, \quad (1.6)$$

$$\tilde{C}(Y,Z)\,U = \widetilde{K}(Y,Z)U - \frac{\tilde{r}}{n(n-1)}\{g(Z,U)Y - g(Y,U)Z\}, \quad (1.7)$$

where$\widetilde{K}, \tilde{S}, \tilde{Q}$ and $\tilde{r}$ are Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature with respect to generalized Tanaka-Webster connection.

2. Preliminaries

An n-dimensional differentiable manifold M_n is called a Lorentzian Para-Sasakian (briefly LP-Sasakian) manifold if it admits a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g which satisfies[4]

$$\phi^2 Y = Y + \eta(Y)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \quad \eta(\phi Y) = 0, \quad \phi\xi = 0, \tag{2.2}$$

$$g(\phi Y, \phi Z) = g(Y, Z) + \eta(Y)\eta(Z), \tag{2.3}$$

$$g(Y, \phi Z) = g(\phi Y, Z), \tag{2.4}$$

$$\nabla_Y \xi = \phi Y, \quad \eta(Y) = g(Y, \xi), \tag{2.5}$$

$$(\nabla_Y \phi)Z = g(Y, Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi, \tag{2.6}$$

for all $Y, Z \in \chi(M_n)$, where ∇ is the covariant derivative with Lorentzian metric g.

Let us put $\omega(Y, Z) = g(Y, \phi Z)$. (2.7)

Also, since the vector field η is closed in LP-Sasakian manifold, we get

$$(\nabla_Y \eta)(Z) = \omega(Y, Z), \qquad \omega(Y, \xi) = 0. \tag{2.8}$$

In addition to the above, the following relations also hold in LP-Sasakian manifolds:

$$\eta(K(Y, Z)U) = g(Z, U)\eta(Y) - g(Y, U)\eta(Z), \tag{2.9}$$

$$K(Y, Z)\xi = \eta(Z)Y - \eta(Y)Z, \tag{2.10}$$

$$K(\xi, Y)Z = g(Y, Z)\xi - \eta(Z)Y, \tag{2.11}$$

$$K(\xi, Y)\xi = \eta(Y)\xi + Y, \tag{2.12}$$

$$S(Y, \xi) = (n-1)\eta(Y), \tag{2.13}$$

$$S(\phi Y, \phi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z), \tag{2.14}$$

where$K$ and S are curvature tensor and Ricci tensor in LP-Sasakian manifold with respect to Riemannian connection.

In consequence of (1.1), (2.5), (2.7) and (2.8), it gives

$$\widetilde{\nabla}_Y Z = \nabla_Y Z + g(Y, \phi Z)\xi - \eta(Z)\phi Y + \eta(Y)\phi Z. \quad (2.15)$$

Definition2.1:A LP-Sasakian manifold M_n is said to be an η-Einstein manifold if Ricci tensor S is of the form

$$S(Y, Z) = \gamma g(Y, Z) + \mu\eta(Y)\eta(Z) \quad (2.16)$$

for any vector fields Y, Z where γ and μ are functions on M_n.

3.Curvature properties ofLP-Sasakian manifold with respect to generalized Tanaka-Webster connection

Let us suppose the vector fields Y, Z and U are orthogonal to ξ.

Then, the curvature tensor $\widetilde{K}$ with respect to the connection $\widetilde{\nabla}$is given as [1]

$$\widetilde{K}(Y, Z)U = K(Y, Z)U + 3g(Z, \phi U)\phi Y - 3g(Y, \phi U)\phi Z. \quad (3.1)$$

Also, Ricci tensor $\tilde{S}$,scalar curvature $\tilde{r}$, Ricci operator $\tilde{Q}$ with respect to the connection $\widetilde{\nabla}$ respectively are given by

$$\tilde{S}(Y, Z) = S(Y, Z) - 3g(Y, Z) - 3\eta(Y)\eta(Z), \quad (3.2)$$

$$\tilde{r} = r - 3(n - 1), \quad (3.3)$$

$$\tilde{Q}Y = QY - 3Y - 3\eta(Y)\xi. \quad (3.4)$$

Theorem3.1: The m-projective and projective curvature tensors of LP-Sasakian manifold M_n with respect to generalized Tanaka-Webster connection, provided vector fields are orthogonal to ξ, are linearly dependent if and only if $M_n$is an η-Einstein manifold.

Proof: We consider, $\widetilde{W}^*(Y, Z)U = \beta\tilde{P}(Y, Z)U,$ (3.5)

whereβ is any non-zero constant.

Using (1.5) and (1.6) in the above relation, we get

$$(1 - \beta)\widetilde{K}(Y, Z)U = \frac{1}{2(n - 1)}\{\tilde{S}(Z, U)Y - \tilde{S}(Y, U)Z + g(Z, U)\tilde{Q}Y$$
$$-g(Y, U)\tilde{Q}Z\} - \frac{1}{n-1}\beta\{\tilde{S}(Z, U)Y - \tilde{S}(Y, U)Z\}. \quad (3.6)$$

Taking inner product with respect to V on both sides of (3.6) and using (3.1), (3.2) and (3.4), we obtain

$(1-\beta)\{g(K(Y,Z)U,V)+3g(Z,\phi U)g(\phi Y,V)-3g(Y,\phi U)g(\phi Z,V)\}=$
$\frac{(1-2\beta)}{2(n-1)}\{S(Z,U)g(Y,V)-3g(Z,U)g(Y,V)-3\eta(Z)\eta(U)g(Y,V)-S(Y,U)g(Z,V)+$
3gY,UgZ,V+3ηYηUgZ,V+12(n−1){gZ,UgQY,V−3gZ,UgY,V−3ηYgZ,Ugξ,V−gY,UgQZ,V+3gY ,UgZ,V+3gY,Uη(Z)g(ξ,V)}.(3.7)

Suppose $\{e_1,\dots,e_n\}$ be an orthonormal basis of tangent space at any point of the manifold. Setting $Y=V=e_i$, in the above relation and taking summation over i, $1\le i\le n$, on both sides of (3.7),

$$(1-\beta)\sum_{i=1}^{n}\{g(K(e_i,Z)U,e_i)+3g(Z,\phi U)g(\phi e_i,e_i)-33g(e_i,\phi U)g(\phi Z,e_i)\}$$
$$=\frac{(1-2\beta)}{2(n-1)}\sum_{i=1}^{n}\{S(Z,U)g(e_i,e_i)-3g(Z,U)g(e_i,e_i)-3\eta(Z)\eta(U)g(e_i,e_i)$$
$$-S(e_i,U)g(Z,e_i)+3g(Y,U)g(e_i,e_i)+3\eta(e_i)\eta(U)g(Z,e_i)\}$$
$$+\frac{1}{2(n-1)}\sum_{i=}^{n}\{g(Z,U)g(Qe_i,e_i)-3g(Z,U)g(e_i,e_i)-3\eta(e_i)g(Z,U)g(\xi,e_i)$$
$$-g(e_i,U)g(QZ,e_i)+3g(e_i,U)g(Z,e_i)+3g(e_i,U)\eta(Z)g(\xi,e_i)\}.$$

$$\Rightarrow$$

$$(1-\beta)\{S(Z,U)-3g(\phi Z,\phi U)\}=\frac{(1-2\beta)}{2}\{S(Z,U)-3g(Z,U)-3\eta(Z)\eta(U)\}$$

$+\frac{1}{2(n-1)}\{(r-3n+6)g(Z,U)-S(Z,U)+3\eta(Z)\eta(U)\}.$

$\Rightarrow$

$$S(Z,U)=\frac{(3+r)}{n}g(Z,U)+3\eta(Z)\eta(U) \qquad (3.8)$$

which shows that M_n is an η-Einstein manifold.

First part of the theorem is proved.

In consequence of (1.5), (1.6) and (3.8), we obtain the converse part of the theorem.

Theorem3.2: The necessary and sufficient condition for a LP-Sasakian manifold to be an η-Einstein manifold is that the m-projective curvature tensor $\widetilde{W}^*$ and concircular curvature $\tilde{C}$ tensor with respect to generalized Tanaka- Webster connection $\widetilde{\nabla}$, provided the vector fields are orthogonal to ξ, are linearly dependent.

Proof: Let$\widetilde{W}^*(Y,Z)U = \beta\tilde{C}(Y,Z)U.$ (3.9)

By virtue of (1.6) and (1.7), (3.9) yields

$$2n(n-1)(1-\beta)\widetilde{K}(Y,Z)U = n\{\tilde{S}(Z,U)Y - \tilde{S}(Y,U)Z + g(Z,U)\tilde{Q}Y - g(Y,U)\tilde{Q}Z\} - 2\tilde{r}\{g(Z,U)Y - g(Y,U)Z\}.$$

$\Rightarrow$

$$2n(n-1)(1-\beta)\widetilde{K}(Y,Z)U = nS(Z,U)Y - nS(Y,U)Z - 3n\eta(Z)\eta(U)Y + 3n\eta(Y)\eta(U)Z + ng(Z,U)QY - ng(Y,U)QZ - 3ng(Z,U)\eta(Y)\xi + 3ng(Y,U)\eta(Z)\xi - (2r+6)\{g(Z,U)Y - g(Y,U)Z\}. \quad (3.10)$$

After taking inner product on both sides of (3.10) with respect to V, we put$Y = V = e_i$ and also, taking summation over $i, 1 \le i \le n$ on both sides of the above equation, we find

$$2n(n-1)(1-\beta)\tilde{S}(Z,U) = \sum_{i=1}^{n}\{nS(Z,U)g(e_i,e_i) - nS(e_i,U)g(Z,V) - 3n\eta(Z)\eta(U)g(e_i,e_i) + 3ng(e_i,\xi)\eta(U)g(Z,e_i) + ng(Z,U)S(e_i,e_i) - ng(e_i,U)S(Z,e_i) - 3ng(Z,U)g(e_i,\xi)g(\xi,e_i) + 3ng(e_i,U)\eta(Z)g(\xi,e_i) - (2r+6)\{g(Z,U)g(e_i,e_i) - g(e_i,U)g(Z,e_i)\}.$$

$\Rightarrow$

$$S(Z,U) = \frac{(r+3)}{n}g(Z,U) + \frac{3n}{(n-2)}\eta(Z)\eta(U). \quad (3.11)$$

This shows that M_n is anη-Einstein manifold.

Converse part is obvious from (1.6), (1.7) and (3.11).

Theorem3.3: If a LP-Sasakian manifoldM_n is an n-dimensional ϕ-projectively flat with respect to the generalized Tanaka-Webster connection$\widetilde{\nabla}$, provided vector fields are orthogonal to ξ,thenM_n is an η-Einstein manifold.

Proof: IfM_n is ϕ- protectively flat LP-Sasakian manifold with respect to $\widetilde{\nabla}$, then we get,

$\phi^2(\tilde{P}(\phi Y, \phi Z)\phi U) = O.$ (3.12)

By making use of (2.1) in (3.12) and taking inner product on it, we obtain

$g(\tilde{P}(\phi Y, \phi Z)\phi U, \phi V) = O.$

$\Rightarrow$

$$g(\widetilde{K}(\phi Y, \phi Z)\phi U - \frac{1}{n-1}\{\tilde{S}(\phi Y, \phi Z)\phi U - \tilde{S}(\phi Y, \phi Z)\phi U,\ \phi V) = O. \quad (3.13)$$

$$g\big(\widetilde{K}(\phi Y, \phi Z)\phi U, \phi V\big) - \frac{1}{n-1}\{\tilde{S}(\phi Y, \phi Z)g(\phi U, \phi V) - \tilde{S}(\phi Y, \phi Z)g(\phi U, \phi V)\} = O.$$

Using (3.1) and (3.2) in the above equation, we get

$$g(K(\phi Y, \phi Z)\phi U, \phi V) + 3g(\phi Z, U)g(Y, \phi V) - 3g(\phi Y, U)\, g(Z, \phi V) =$$
$$\frac{1}{n-1}\{S(\phi Z, \phi U)g(\phi Y, \phi V) - 3g(\phi Z, \phi U)g(\phi Y, \phi V) - S(\phi Y, \phi U)g(\phi Z, \phi V)\} +$$
$$3g(\phi Y, \phi U)\, g(\phi Z, \phi V)\}. \quad (3.14)$$

Let $\{e_1,\ e_2,\ \ldots..e_{n-1},\ \xi\}$ be a local orthonormal basis of vector fields in M_n. Using$\{\phi e_1,\ \phi e_2,$ $\ldots..\phi e_{n-1}, \xi\}$ that is also a local orthonormal basis, if we put $X = U = e_i$in the above equation and sum up with respect to i, then

$$\sum_{i-1}^{n-1} g(K(\phi e_i, \phi Z)\phi U, \phi e_i) + \sum_{i-1}^{n-1} 3g(\phi Z, U)g(e_i, \phi e_i) - \sum_{i-1}^{n-1} 3g(\phi e_i, U)g(Z, \phi e_i)$$
$$= \frac{1}{n-1}\sum_{i-1}^{n-1}\{S(\phi Z, \phi U)g(e_i, \phi e_i) - 3g(\phi Z, \phi U)g(\phi e_i, \phi e_i)$$
$$- S(\phi U, \phi e_i)g(\phi Z, \phi e_i) + 3g(\phi e_i, \phi U)g(\phi Z, \phi e_i)\}.$$

$\Rightarrow$

$$S(\phi Z, \phi U) - 3g(Z, U) = \frac{1}{n-1}\{S(\phi Z, \phi U)(n+1) + 3g(\phi Z, \phi U)(n+1) - S(\phi Z, \phi U) +$$
$$3g(\phi Z, \phi U)\}. \quad (3.15)$$

With the help of (2.14), the above equation becomes

$S(Z, U) = -(6n + 3)\, g(Z, U) - (4n + 5)\eta(Z)\eta(U),$

which shows that M_n is η –Einstein Manifold.

Hence the proof is over.

Theorem3.4: If M_n is n-dimensional ϕ-concircularly flat LP-Sasakian manifold with respect to generalized Tanaka-Webster connection, provided the vector fields are orthogonal to ξ then M_n is an η-Einstien manifold.

Proof IfM_n is ϕ-concircularly flat LP-Sasakian manifold with respect to generalized Tanaka-Webster connection$\widetilde{\nabla}$,then we have

$$\phi^2(\tilde{C}(\phi Y,\phi Z)\phi U) = 0. \quad (3.16)$$

Taking inner product on both sides of (3.16) with with respect to ϕV and by using (2.1) on it , we find

$$g(\tilde{C}(\phi Y,\phi Z)\phi U,\phi V\ ) = 0. \quad (3.17)$$

With the help of (1.7), the above equation can be written as

$$g(\widetilde{K}(\phi Y,\phi Z)\phi U,\phi V) - \frac{\tilde{r}}{n(n-1)}\{g(\phi Z,\phi U)g(\phi Y,\phi V) - g(\phi Y,\phi U)g(\phi Z,\phi V)\} = 0.$$

$\Rightarrow$

$$g(K(\phi Y,\phi Z)\phi U,\phi V) + 3g(\phi Z,U)g(Y,\phi V) - 3g(\phi Y,U)g(Z,\phi V) =$$
$$\frac{r-3(n-1)}{n(n-1)}\{g(\phi Z,\phi U)g(\phi Y,\phi V) - g(\phi Y,\phi U)g(\phi Z,\phi V)\}. \quad (3.18)$$

Let $\{e_1, e_2, \ldots..e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in Mn. Using that $\{\phi e_1, \phi e_2, \ldots..\phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = U = e_i$in the above equation and sum up with respect to i, then

$$\sum_{i-1}^{n-1} g(K(\phi e_i,\phi Z)\phi U,\phi e_i) + \sum_{i-1}^{n-1} 3g(\phi Z.U)g(\phi e_i,\phi e_i) - \sum_{i=1}^{n-1} 3g(\phi e_i,U)g(Z,\phi e_i) =$$
$$\frac{r-3(n-1)}{n-1}\sum_{i-1}^{n-1}\{g(\phi Z,\phi U)g(\phi e_i,\phi e_i) - \sum_{i-1}^{n-1} g(\phi e_i,\phi U)g(\phi Z,\phi e_i)\}.$$

$\Rightarrow$

$$S(\phi Z,\phi U) - 3g(Z,U) = \frac{\{r-3(n-1)\}n}{n-1} g(\phi Z,\phi U). \quad (3.19)$$

With the help of (2.3) and (2.14), the above equation reduces to the following

$$S(Z,U) = \frac{r}{n-1}\ g(Z,U) + \frac{(r-n-2)}{n-1}\eta(Z)\eta(U).$$

Hence the proof is completed.

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