Common Fixed Point Theorems for Self Mappings on a Complete Vector S-metric Space

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Abstract
S-metric space was introduced by Sedghi et al. [5] in 2012. We derive some common fixed point results for self-mappings on vector valued complete S-metric space. In support of our results, we also give some examples.
Keywords: Vector lattice, Vector metric space, Vector S-metric space.

1 Introduction

Fixed point theory is amongst the crucial mathematical theory with applications in various branches of science. Banach contraction principle was derived first by S. Banach [2] in 1922. It has a vital role in fixed point theory and became very famous due to iterations used in the theorem. The evaluation of fixed points of mappings satisfying many contractive conditions is at the center of research work and several vital results have been established by many authors. Over the last few years, several researchers have devoted themselves to define many variations of metric space. We give below some definitions and results which will help in proving our main results for vector S-metric spaces.

Definition 1.1[4] On a set $\complement$, a relation $\preceq$ is a partial order if it follows the conditions stated below:

$(a)\eta_1 \preceq \eta_1$ (reflexive)

$(b)\eta_1 \preceq \eta_2$ and $\eta_2 \preceq \eta_1$ implies $\eta_1 = \eta_2$ (anti-symmetry)

$(c)\eta_1 \preceq \eta_2$ and $\eta_2 \preceq \eta_3$ implies $\eta_1 \preceq \eta_3$ (transitivity)

$\forall \eta_1, \eta_2, \eta_3 \in \complement$. The set $\complement$ with partial order $\preceq$ is known as partially ordered set (poset).

A partially ordered set $(\complement, \preceq)$ is called linearly ordered if for $\eta_1, \eta_2 \in \complement$, we have either $\eta_1 \preceq \eta_2$ or $\eta_2 \preceq \eta_1$.

Definition 1.2[4] Let $\complement$ be linear space which is real and $(\complement, \preceq)$ be a poset. Then the poset $(\complement, \preceq)$ is said to be an ordered linear space if it follows the properties mentioned below:

$$(a)p_1 \preceq p_2 \Longrightarrow p_1 + p_3 \preceq p_2 + p_3$$

$(b)p_1 \preceq p_2 \Longrightarrow \omega p_1 \preceq \omega p_2$ $\forall p_1, p_2, p_3 \in \complement$ and $\omega > 0$.

Definition 1.3[4] A poset is called lattice if each set with two elements has an infimum and a supremum.

Definition 1.4[4] An ordered linear space where the ordering is lattice is called vector lattice.

Definition 1.5[4] A vector lattice V is called Archimedean if $inf\{\frac{1}{m}\vartheta\} = 0$ for every $\vartheta \in V^+$ where

$$V^+ = \{\vartheta \in V : \vartheta \geq 0\}.$$

Definition 1.6[3] Let V be a vector lattice and $\Re$ be a nonvoid set. A function $d: \Re \times \Re \to V$ is called vector metric on $\Re$ if it follows the conditions stated below:

(a) $d(\hbar_1, \hbar_2) = 0$ iff $\hbar_1 = \hbar_2$

(b) $d(\hbar_1, \hbar_2) \preceq d(\hbar_1, \hbar_3) + d(\hbar_3, \hbar_2) \forall \hbar_1, \hbar_2, \hbar_3 \in \Re$

The triplet $(\Re, d, V)$ is called vector metric space.

Definition 1.7[6] Let $\Re$ be a nonvoid set. A function $S: \Re \times \Re \times \Re \to [0, \infty)$ is called S-metric on $\Re$ if it follows the conditions stated below:

(a) $S(\flat_1, \flat_2, \flat_3) \geq 0$,

(b) $S(\flat_1, \flat_2, \flat_3) = 0$ iff $\flat_1 = \flat_2 = \flat_3$,

(c) $S(\flat_1, \flat_2, \flat_3) \preceq S(\flat_1, \flat_2, \alpha) + S(\flat_2, \flat_2, \alpha) + S(\flat_3, \flat_3, \alpha)$,

for all $\flat_1, \flat_2, \flat_3, \alpha \in \Re$.

The pair $(\Re, S)$ is called S-metric space .

Now, we define vector valued S-metric space as follows:

Definition 1.8 Let V be a vector lattice and $\Re$ be a nonvoid set. A function $S: \Re \times \Re \times \Re \to V$ is called vector S-metric on $\Re$ that satisfies the conditions mentioned below:

(a) $S(\flat_1, \flat_2, \flat_3) \geq 0$,

(b) $S(\flat_1, \flat_2, \flat_3) = 0$ iff $\flat_1 = \flat_2 = \flat_3$,

(c) $S(\flat_1, \flat_2, \flat_3) \preceq S(\flat_1, \flat_2, \alpha) + S(\flat_2, \flat_2, \alpha) + S(\flat_3, \flat_3, \alpha)$,

for all $\flat_1, \flat_2, \flat_3, \alpha \in \Re$.

The triplet $(\Re, S, V)$ is called vector S-metric space.

Example 1.9 Let $\Re$ be a nonvoid set and V be a vector lattice. A function $S: \Re \times \Re \times \Re \to V$ is defined by

$$S(\flat_1, \flat_2, \flat_3) = |(\flat_1, \flat_3)| + |(\flat_2, \flat_3)| \quad \forall \flat_1, \flat_2, \flat_3 \in \Re$$

then the triplet $(\Re, S, V)$ is vector S-metric space.

Definition 1.10 A sequence $\langle \vartheta_n \rangle$ in a vector S-metric space $(\Re, S, V)$ is called V-convergent to some $\vartheta \in V$ if there is a sequence $\langle \mu_n \rangle$ in V satisfying $\mu_n \downarrow 0$ and $S(\vartheta_n, \vartheta_n, \vartheta) \leq \mu_n$ and denote it by $\vartheta_n \xrightarrow{S,V} \vartheta$.

Definition 1.11 A sequence $\langle \vartheta_n \rangle$ in a vector S-metric space $(\Re, S, V)$ is known as V-Cauchy sequence if $\exists$ $\langle \mu_n \rangle \in V$ satisfying $\mu_n \downarrow 0$ and $S(\vartheta_n, \vartheta_n, \vartheta_{n+q}) \leq \mu_n \forall q$ and n.

Definition 1.12 A vector S-metric space $(\mathfrak{R}, S, V)$ is called V-complete if all V-Cauchy sequence is V-convergent to a limit in $\mathfrak{R}$.

Lemma 1.13[6] For a vector S-metric space $(\mathfrak{R}, S, V)$,

$$S(\vartheta,\vartheta,\mu) = S(\mu,\mu,\vartheta) \qquad \forall \mu,\vartheta \in \mathfrak{R}.$$

Proof. Using the condition (c) of definition (1.8), we have

$$S(\vartheta,\vartheta,\mu) \leq S(\vartheta,\vartheta,\vartheta) + S(\vartheta,\vartheta,\vartheta) + S(\mu,\mu,\vartheta) \tag{1}$$

$$= S(\mu,\mu,\vartheta)$$

$$S(\mu,\mu,\vartheta) \leq S(\mu,\mu,\mu) + S(\mu,\mu,\mu) + S(\vartheta,\vartheta,\mu) \tag{2}$$

$$= S(\vartheta,\vartheta,\mu)$$

By (1) and (2), we get $S(\vartheta,\vartheta,\mu) = S(\mu,\mu,\vartheta)$.

2 Main Results

Lemma 2.1 Let $(\mathfrak{R}, S, V)$ be a vector S-metric space which is complete and V-Archimedean. Let a sequence $\langle \hbar_\flat \rangle$ be in $\mathfrak{R}$ such that

$$S(\hbar_\flat,\hbar_\flat,\hbar_{\flat+1}) \preceq \alpha S(\hbar_{\flat-1},\hbar_{\flat-1},\hbar_\flat) \qquad \forall \flat \in \mathbb{N} (3)$$

where $\alpha \in [0,1)$. Then $\langle \hbar_\flat \rangle$ is a V- Cauchy sequence in $\mathfrak{R}$.

Proof. Using (3), we get

$$S(\hbar_\flat,\hbar_\flat,\hbar_{\flat+1}) \preceq \alpha S(\hbar_{\flat-1},\hbar_{\flat-1},\hbar_\flat) \preceq \alpha^2 S(\hbar_{\flat-2},\hbar_{\flat-2},\hbar_{\flat-1}) \preceq \cdots \preceq \alpha^\flat S(\hbar_0,\hbar_0,\hbar_1)$$

So, for $\flat > \ell$, we have

$$S(\hbar_\ell,\hbar_\ell,\hbar_\flat) \preceq 2S(\hbar_\ell,\hbar_\ell,\hbar_{\ell+1}) + S(\hbar_\flat,\hbar_\flat,\hbar_{\ell+1})$$

$$= 2S(\hbar_\ell,\hbar_\ell,\hbar_{\ell+1}) + S(\hbar_{\ell+1},\hbar_{\ell+1},\hbar_\flat)$$

$$\preceq 2S(\hbar_\ell,\hbar_\ell,\hbar_{\ell+1}) + 2S(\hbar_{\ell+1},\hbar_{\ell+1},\hbar_{\ell+2}) + S(\hbar_\flat,\hbar_\flat,\hbar_{\ell+2})$$

$$= 2S(\hbar_\ell,\hbar_\ell,\hbar_{\ell+1}) + 2S(\hbar_{\ell+1},\hbar_{\ell+1},\hbar_{\ell+2}) + S(\hbar_{\ell+2},\hbar_{\ell+2},\hbar_\flat)$$

$$\preceq 2S(\hbar_\ell,\hbar_\ell,\hbar_{\ell+1}) + 2S(\hbar_{\ell+1},\hbar_{\ell+1},\hbar_{\ell+2}) + \cdots + S(\hbar_{\flat-1},\hbar_{\flat-1},\hbar_\flat)$$

$$\prec 2S(\hbar_\ell,\hbar_\ell,\hbar_{\ell+1}) + 2S(\hbar_{\ell+1},\hbar_{\ell+1},\hbar_{\ell+2}) + \cdots + 2S(\hbar_{\flat-1},\hbar_{\flat-1},\hbar_\flat)$$

$$\prec 2(\alpha^\ell + \alpha^{\ell+1} + \cdots + \alpha^{\flat-1}) S(\hbar_0,\hbar_0,\hbar_1)$$

$$\prec 2\alpha^\ell (1 + \alpha + \alpha^2 + \cdots) S(\hbar_0,\hbar_0,\hbar_1)$$

$$\prec 2\frac{\alpha^\ell}{1-\alpha} S(\hbar_0,\hbar_0,\hbar_1) \downarrow 0 \qquad \ell \to \infty.$$

Thus $\langle \hbar_\flat \rangle$ is a V-Cauchy sequence.

Theorem 2.2 Let $(\mathfrak{R}, S, V)$ be a vector S-metric space which is complete and V- Archimedean. Let $K: \mathfrak{R} \to \mathfrak{R}$ be a continuous mapping and a map $f: \mathfrak{R} \to \mathfrak{R}$ which commutes with K. Suppose the conditions given below are satisfied;

$$(a)\ f(\mathfrak{R}) \subseteq K(\mathfrak{R})$$

(b) $S(f\hbar, f\hbar, f\vartheta) \preceq qU(\hbar, \hbar, \vartheta)$ for all $\hbar, \vartheta \in \mathfrak{R}$ where $q \in [0, \frac{1}{3})$ is a constant and

$$U(\hbar, \hbar, \vartheta) \in \{S(K\hbar, K\hbar, K\vartheta), S(K\hbar, K\hbar, f\hbar), S(K\vartheta, K\vartheta, f\vartheta), S(K\hbar, K\hbar, f\vartheta), S(K\vartheta, K\vartheta, f\hbar)\}$$

(c) $K(\mathfrak{R})$ or $f(\mathfrak{R})$ is V- complete as a subspace of $\mathfrak{R}$. Then, prove that K and f have a common fixed point which is unique .

Proof Fix arbitrary $\vartheta_0 \in \mathfrak{R}$, so we can take sequence $\langle \hbar_\flat \rangle$ in $\mathfrak{R}$ such that

$$\hbar_\flat = f\vartheta_\flat = K\vartheta_{\flat+1}\flat \geq 0.$$

Then

$$S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) = S(f\vartheta_\flat, f\vartheta_\flat, f\vartheta_{\flat+1}) \preceq qU(\vartheta_\flat, \vartheta_\flat, \vartheta_{\flat+1})(4)$$

Where

$$U(\vartheta_\flat, \vartheta_\flat, \vartheta_{\flat+1}) \in \{S(K\vartheta_\flat, K\vartheta_\flat, K\vartheta_{\flat+1}), S(K\vartheta_\flat, K\vartheta_\flat, f\vartheta_\flat), S(K\vartheta_{\flat+1}, K\vartheta_{\flat+1}, f\vartheta_{\flat+1}),$$
$$S(K\vartheta_\flat, K\vartheta_\flat, f\vartheta_{\flat+1}), S(K\vartheta_{\flat+1}, K\vartheta_{\flat+1}, f\vartheta_\flat)\}$$
$$= \{S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat), S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat), S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}),$$
$$S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_{\flat+1}), S(\hbar_\flat, \hbar_\flat, \hbar_\flat)\}$$
$$= \{S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat), S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}), S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_{\flat+1}), 0\}$$

The possible four cases are:

$$(a)\ S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) \preceq qS(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat).$$

(b) $S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) \preceq qS(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1})$

and so

$$S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) = 0.$$

(c) $S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) \preceq qS(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_{\flat+1})$

$$\preceq 2qS(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat) + qS(\hbar_{\flat+1}, \hbar_{\flat+1}, \hbar_\flat)$$

and

$$S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) \preceq \frac{2q}{(1-q)} S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat).$$

$(d) S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) \preceq q.0 = 0$

and so

$$S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) = 0.$$

Thus $S(\hbar_\flat, \hbar_\flat, \hbar_{\flat+1}) \preceq \sigma S(\hbar_{\flat-1}, \hbar_{\flat-1}, \hbar_\flat)$ where $\sigma \in \{q, \frac{2q}{(1-q)}\} < 1$.

Since V is Archimedean, by lemma (2.1) $\langle \hbar_\flat \rangle$ is a V-Cauchy sequence and range of f is contained in the range of K and atleast one range is V-complete, there exist $\hbar \in K(\mathfrak{R})$ such that $K\vartheta_{\flat+1} \xrightarrow{S,V} \hbar$. Hence there exist a sequence $\langle \alpha_\flat \rangle$ in V such that $\alpha_\flat \downarrow 0$ and

$$S(K\vartheta_\flat, K\vartheta_\flat, \hbar) \preceq \alpha_\flat.$$

Thus

$$\hbar_\flat = f\vartheta_\flat = K\vartheta_{\flat+1} \xrightarrow{S,V} \hbar. \tag{5}$$

We prove that

$$K\hbar = f\hbar = \hbar$$

Now,

$$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + S(f\hbar, f\hbar, fK\vartheta_\flat)$$

$$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + S(fK\vartheta_\flat, fK\vartheta_\flat, f\hbar) \tag{6}$$

Also, we have

$$S(fK\vartheta_\flat, fK\vartheta_\flat, f\hbar) \preceq qU(K\vartheta_\flat, K\vartheta_\flat, \hbar)$$

Then (5) becomes

$$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + qU(K\vartheta_\flat, K\vartheta_\flat, \hbar) \tag{7}$$

where

$$U(K\vartheta_\flat, K\vartheta_\flat, \hbar) \in \{S(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar), S(K^2\vartheta_\flat, K^2\vartheta_\flat, fK\vartheta_\flat), S(K\hbar, K\hbar, f\hbar),$$
$$S(K^2\vartheta_\flat, K^2\vartheta_\flat, f\hbar), S(K\hbar, K\hbar, fK\vartheta_\flat)\} \tag{8}$$

Since f commutes with K and by using continuity of K, we get

$$fK\vartheta_\flat = Kf\vartheta_\flat \xrightarrow{S,V} K\hbar$$

and by using (5)

$$K^2\vartheta_{\flat+1} \xrightarrow{S,V} K\hbar,$$

then there exist a sequence $\langle\alpha_\flat\rangle$ and $\langle\beta_\flat\rangle$ in V such that $\alpha_\flat \downarrow 0$ and $\beta_\flat \downarrow 0$, then we have

$$S(K\hbar, K\hbar, fK\vartheta_\flat) \preceq \alpha_\flat$$

and

$$S(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar) \preceq \beta_\flat.$$

By (7) and (8), we have the following cases:

(i) $$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + qS(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar)$$
$$\preceq 2\alpha_\flat + q\beta_\flat$$

(ii) $$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + qS(K^2\vartheta_\flat, K^2\vartheta_\flat, fK\vartheta_\flat)$$
$$\preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + q[2S(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar) + S(fK\vartheta_\flat, fK\vartheta_\flat, K\hbar)]$$

$$= 2S(K\hbar, K\hbar, fK\vartheta_\flat) + 2qS(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar) + qS(K\hbar, K\hbar, fK\vartheta_\flat)$$

$$\preceq (2+q)\alpha_\flat + 2q\beta_\flat$$

(iii) $$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + qS(K\hbar, K\hbar, f\hbar)$$

$$(1-q)S(K\hbar, K\hbar, f\hbar) \preceq 2\alpha_\flat$$

$$S(K\hbar, K\hbar, f\hbar) \preceq \frac{2\alpha_\flat}{(1-q)}$$

(iv) $$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + qS(K^2\vartheta_\flat, K^2\vartheta_\flat, f\hbar)$$

$$\preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + 2qS(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar) + qS(f\hbar, f\hbar, K\hbar)$$

$$= 2S(K\hbar, K\hbar, fK\vartheta_\flat) + 2qS(K^2\vartheta_\flat, K^2\vartheta_\flat, K\hbar) + qS(K\hbar, K\hbar, f\hbar)$$

$$(1-q)S(K\hbar, K\hbar, f\hbar) \preceq 2\alpha_\flat + 2q\beta_\flat$$

$$S(K\hbar, K\hbar, f\hbar) \preceq \frac{2\alpha_\flat + 2q\beta_\flat}{(1-q)}$$

(v) $$S(K\hbar, K\hbar, f\hbar) \preceq 2S(K\hbar, K\hbar, fK\vartheta_\flat) + qS(K\hbar, K\hbar, fK\vartheta_\flat)$$

$$\preceq (2+q)S(K\hbar, K\hbar, fK\vartheta_\flat)$$

$$\preceq (2+q)\alpha_\flat$$

$$\preceq 3\alpha_\flat$$

In the last inequality of each case, the infimum on the right hand side is 0. So we get

$$S(K\hbar, K\hbar, f\hbar) = 0.$$

This implies

$$K\hbar = f\hbar \tag{9}$$

So

$$S(K\hbar, K\hbar, \hbar) \preceq 2S(K\hbar, K\hbar, f\vartheta_\flat) + S(\hbar, \hbar, f\vartheta_\flat)$$

$$= 2S(K\hbar, K\hbar, f\vartheta_\flat) + S(f\vartheta_\flat, f\vartheta_\flat, \hbar)$$

$$\preceq S(f\vartheta_\flat, f\vartheta_\flat, \hbar) + 2S(f\hbar, f\hbar, f\vartheta_\flat)$$

$$= S(f\vartheta_\flat, f\vartheta_\flat, \hbar) + 2S(f\vartheta_\flat, f\vartheta_\flat, f\hbar)$$

$$S(K\hbar, K\hbar, \hbar) \preceq S(f\vartheta_\flat, f\vartheta_\flat, \hbar) + 2qU(\vartheta_\flat, \vartheta_\flat, \hbar) \tag{10}$$

where

$$U(\vartheta_\flat, \vartheta_\flat, \hbar) \in \{S(K\vartheta_\flat, K\vartheta_\flat, K\hbar), S(K\vartheta_\flat, K\vartheta_\flat, f\vartheta_\flat), S(K\hbar, K\hbar, f\hbar), S(K\vartheta_\flat, K\vartheta_\flat, f\hbar), S(K\hbar, K\hbar, f\vartheta_\flat)\}$$

$$= \{S(K\vartheta_\flat, K\vartheta_\flat, K\hbar), S(K\vartheta_\flat, K\vartheta_\flat, f\vartheta_\flat), 0, S(K\hbar, K\hbar, f\vartheta_\flat)\} \tag{11}$$

there exist a sequence $\langle d_n \rangle$ and $\langle g_n \rangle$ in V such that $d_n \downarrow 0$ and $g_n \downarrow 0$, then we have

$$S(K\vartheta_{\flat}, K\vartheta_{\flat}, \hbar) \preceq d_n$$

and

$$S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) \preceq g_n.$$

Using (10) and (11), we have the following cases:

(i) $S(K\hbar, K\hbar, \hbar) \preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2qS(K\vartheta_{\flat}, K\vartheta_{\flat}, K\hbar)$

$$\preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2q[2S(K\vartheta_{\flat}, K\vartheta_{\flat}, \hbar) + S(K\hbar, K\hbar, \hbar)]$$

$(1-2q)S(K\hbar, K\hbar, \hbar) \preceq g_n + 4qd_n$

$$S(K\hbar, K\hbar, \hbar) \preceq \frac{g_n + 4qd_n}{(1-2q)}$$

(ii) $S(K\hbar, K\hbar, \hbar) \preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2qS(K\vartheta_{\flat}, K\vartheta_{\flat}, f\vartheta_{\flat})$

$$\preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2q[2S(K\vartheta_{\flat}, K\vartheta_{\flat}, \hbar) + S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar)]$$

$$= S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2q[2S(K\vartheta_{\flat}, K\vartheta_{\flat}, \hbar) + S(\hbar, \hbar, f\vartheta_{\flat})]$$

$S(K\hbar, K\hbar, \hbar) \preceq g_n + 2q(2d_n + g_n)$

$$\preceq g_n(1+2q) + 4qd_n$$

(iii) $S(K\hbar, K\hbar, \hbar) \preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2q.0$

$S(K\hbar, K\hbar, \hbar) \preceq g_n$

(iv) $S(K\hbar, K\hbar, \hbar) \preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2qS(K\hbar, K\hbar, f\vartheta_{\flat})$

$$\preceq S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2q[2S(K\hbar, K\hbar, \hbar) + S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar)]$$

$$= S(f\vartheta_{\flat}, f\vartheta_{\flat}, \hbar) + 2q[2S(K\hbar, K\hbar, \hbar) + S(\hbar, \hbar, f\vartheta_{\flat})]$$

$(1-4q)S(K\hbar, K\hbar, \hbar) \preceq g_n + 2qg_n$

$$S(K\hbar, K\hbar, \hbar) \preceq \frac{(1+2q)}{(1-4q)} g_n.$$

In the last inequality of each case, the infimum on the right hand side is 0. So we get$S(K\hbar, K\hbar, \hbar) = 0$.

So

$$K\hbar = \hbar.$$

From (9)

$$K\hbar = f\hbar = \hbar$$

So, f and K have common fixed point $\hbar$.

If K and f have another common fixed point μ_1 then

$$K\mu_1 = f\mu_1 = \mu_1.$$

From hypothesis (b)

$$S(\hbar,\hbar,\mu_1)=S(f\hbar,f\hbar,f\mu_1)\preceq qU(\hbar,\hbar,\mu_1)$$

where

$$U(\hbar,\hbar,\mu_1)\in\{S(K\hbar,K\hbar,K\mu_1),S(K\hbar,K\hbar,f\hbar),S(K\mu_1,K\mu_1,f\mu_1),S(K\hbar,K\hbar,f\mu_1),$$
$$S(K\mu_1,K\mu_1,f\hbar)\}$$
$$=\{0,S(\hbar,\hbar,\mu_1)\}.$$

Thus

$$S(\hbar,\hbar,\mu_1)=0.$$

So

$$\hbar=\mu_1.$$

Hence f and K have a common fixed point $\hbar$ that is unique.

Corollary 2.3 Let $(\mathfrak{R},S,V)$ be a vector S-metric space which is complete and V- Archimedean. Let $K:\mathfrak{R}\to\mathfrak{R}$ be a map which is continuous and $f:\mathfrak{R}\to\mathfrak{R}$ be a map which commutes with K. Also let f and K satisfy$f(\mathfrak{R})\subseteq K(\mathfrak{R})$ and $S(f\hbar,f\hbar,f\vartheta)\preceq qS(K\hbar,K\hbar,K\vartheta)$ for all $q\in[0,\frac{1}{3})$ and $\hbar,\vartheta\in\mathfrak{R}$. Then K and f have common fixed point which is unique.

Theorem 2.4 Let $(\mathfrak{R},S,V)$ be a vector S-metric space which is complete and V-Archimedean. Suppose a map $K^2:\mathfrak{R}\to\mathfrak{R}$ is continuous and a map $f:\mathfrak{R}\to\mathfrak{R}$ which commutes with K. Suppose the conditions given below are satisfied;

$$(a)\ fK(\mathfrak{R})\subseteq K^2(\mathfrak{R})$$

(b) $S(f\hbar,f\hbar,f\vartheta)\preceq qU(\hbar,\hbar,\vartheta)$ for all $\hbar,\vartheta\in\mathfrak{R}$ where $q\in[0,\frac{1}{3})$ is a constant and

$$U(\hbar,\hbar,\vartheta)\in\{S(K\hbar,K\hbar,K\vartheta),S(K\hbar,K\hbar,f\hbar),S(K\vartheta,K\vartheta,f\vartheta),\frac{1}{3}[S(K\hbar,K\hbar,f\vartheta)+S(K\vartheta,K\vartheta,f\hbar)]\}$$

(c) $K(\mathfrak{R})$ or $f(\mathfrak{R})$ as a subspace of $\mathfrak{R}$ which is V-complete.

Then prove that K and f have a common fixed point which is unique.

Proof. Fix arbitrary $\vartheta_0\in K(\mathfrak{R})$, so we can take sequence $\langle\hbar_\flat\rangle$ in $K(\mathfrak{R})$ such that

$$\hbar_\flat=f\vartheta_\flat=K\vartheta_{\flat+1}\flat\geq 0.$$

Now

$$K\hbar_{\flat+1}=Kf\vartheta_{\flat+1}=fK\vartheta_{\flat+1}=f\hbar_\flat=p_\flat\flat\geq 0.$$

It can be shown as in theorem(2.2) that $\langle p_\flat\rangle$ is a V-Cauchy sequence and converge to $p\in\mathfrak{R}$ and

$$K^2p=fKp$$

$$\lim_{\flat\to\infty}Kf\vartheta_\flat=\lim_{\flat\to\infty}K\hbar_\flat=\lim_{\flat\to\infty}p_{\flat-1}=p. \qquad (12)$$

It follows that

$$\lim_{\flat\to\infty} K^4\vartheta_\flat = \lim_{\flat\to\infty} K^3(K\vartheta_\flat)$$

$$= \lim_{\flat\to\infty} K^3(\hbar_{\flat-1})$$

$$= \lim_{\flat\to\infty} K^2(K\hbar_{\flat-1})$$

$$= \lim_{\flat\to\infty} K^2(p_{\flat-2})$$

$$= K^2p, \text{because } K^2 \text{ is continuous.} \quad (13)$$

So we get

$$S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_\flat) + S(fKp, fKp, fK^3\vartheta_\flat)$$

$$= 2S(K^2p, K^2p, fK^3\vartheta_\flat) + S(fK^3\vartheta_\flat, fK^3\vartheta_\flat, fKp)$$

$$S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_\flat) + qU(K^3\vartheta_\flat, K^3\vartheta_\flat, Kp) \quad (14)$$

where

$$U(K^3\vartheta_\flat, K^3\vartheta_\flat, Kp) \in \{S(K^4\vartheta_\flat, K^4\vartheta_\flat, K^2p), S(K^4\vartheta_\flat, K^4\vartheta_\flat, fK^3\vartheta_\flat), S(K^2p, K^2p, fKp),$$

$$\frac{1}{3}[S(K^4\vartheta_\flat, K^4\vartheta_\flat, fKp) + S(K^2p, K^2p, fK^3\vartheta_\flat)]\}. (15)$$

Since K^2 is continuous and by using (12), we get

$$K^3 f\vartheta_\flat \xrightarrow{S,V} K^2p$$

and

$$K^4\vartheta_\flat \xrightarrow{S,V} K^2p$$

then there exist a sequence $\langle\alpha_\flat\rangle$ and $\langle\beta_\flat\rangle$ in V such that $\alpha_\flat \downarrow 0$ and $\beta_\flat \downarrow 0$, then we have

$$S(K^2p, K^2p, fK^3\vartheta_\flat) \preceq \alpha_\flat$$

$$S(K^4\vartheta_\flat, K^4\vartheta_\flat, K^2p) \preceq \beta_\flat.$$

Using (14) and (15), then we have the following:

$Case(i)$ $S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_\flat) + qS(K^4\vartheta_\flat, K^4\vartheta_\flat, K^2p)$

$$\preceq 2\alpha_\flat + q\beta_\flat$$

$Case(ii)$ $S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_\flat) + qS(K^4\vartheta_\flat, K^4\vartheta_\flat, fK^3\vartheta_\flat)$

$$\preceq 2S(K^2p, K^2p, fK^3\vartheta_\flat) + q[2S(K^4\vartheta_\flat, K^4\vartheta_\flat, K^2p) +$$

$$S(fK^3\vartheta_\flat, fK^3\vartheta_\flat, K^2p)]$$

$$= 2S(K^2p, K^2p, fK^3\vartheta_\flat) + q[2S(K^4\vartheta_\flat, K^4\vartheta_\flat, K^2p) +$$

$S(K^2p, K^2p, K^3f\vartheta_b)]$

$\preceq 2\alpha_b + q(2\beta_b + \alpha_b)$

$\preceq (2+q)\alpha_b + 2q\beta_b$

$Case(iii) S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_b) + qS(K^2p, K^2p, fKp)$

$$(1-q)S(K^2p, K^2p, fKp) \preceq 2\alpha_b$$

$$S(K^2p, K^2p, fKp) \preceq \frac{2\alpha_b}{(1-q)}$$

$Case(iv)\ S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_b) + \frac{q}{3}[S(K^4\vartheta_b, K^4\vartheta_b, fKp) + S(K^2p, K^2p, fK^3\vartheta_b)]$

$$S(K^2p, K^2p, fKp) \preceq 2S(K^2p, K^2p, fK^3\vartheta_b) + \frac{q}{3}[2S(K^4\vartheta_b, K^4\vartheta_b, K^2p) + S(fKp, fKp, K^2p) + S(K^2p, K^2p, fK^3\vartheta_b)]$$

$$= 2S(K^2p, K^2p, fK^3\vartheta_b) + \frac{q}{3}[2S(K^4\vartheta_b, K^4\vartheta_b, K^2p) + S(K^2p, K^2p, fKp) + S(K^2p, K^2p, fK^3\vartheta_b)]$$

$$(1-\frac{q}{3})S(K^2p, K^2p, fKp) \preceq 2\alpha_b + \frac{q}{3}[2\beta_b + \alpha_b]$$

$$(3-q)S(K^2p, K^2p, fKp) \preceq 6\alpha_b + q[2\beta_b + \alpha_b]$$

$$S(K^2p, K^2p, fKp) \preceq \frac{(6+q)\alpha_b + 2q\beta_b}{(3-q)}.$$

In the last inequality of each case, the infimum on the right hand side is 0. So

$$S(K^2p, K^2p, fKp) = 0.$$

that is

$$K^2p = fKp.$$

Putting in $S(f\hbar, f\hbar, f\vartheta) \preceq qU(\hbar, \hbar, \vartheta)$, $\hbar = fKp, \vartheta = Kp$ then we get $f(fKp) = fK$, we get $f(fKp) = fKp$. Because $K^2p = fKp$ i.e. $K(Kp) = f(Kp)$, we have $K(fKp) = fK^2p = f(fKp) = fKp$. So f and K have a common fixed point fKp .

If f and K have a common fixed point $f_1K_1p_1$ then

$$K(f_1K_1p_1) = f(f_1K_1p_1) = f_1K_1p_1$$

Now

$$S(fKp, fKp, f_1K_1p_1) = S(f(fKp), f(fKp), f(f_1K_1p_1)) \preceq qU(fKp, fKp, f_1K_1p_1)$$

where

$U(fKp, fKp, f_1K_1p_1) \in \{S(K(fKp), K(fKp), K(f_1K_1p_1)), S(K(fKp), K(fKp), f(fKp)),$

$$S(K(f_1K_1p_1), K(f_1K_1p_1), f(f_1K_1p_1)), S(K(fKp), K(fKp), f(f_1K_1p_1))),$$
$$S(K(f_1K_1p_1), K(f_1K_1p_1), f(fKp))\}$$
$$\in \{0, S(fKp, fKp, f_1K_1p_1)\} \tag{16}$$

Thus

$$S(fKp, fKp, f_1K_1p_1) = 0.$$

So

$$fKp = f_1K_1p_1.$$

Hence f and K have a common fixed point fKp which is unique.

Example 2.5 Let $V = \mathbb{R}_+^2$ with coordinatewise ordering and $\mathfrak{R} = \mathbb{R}$

$$S(\hbar, \vartheta, \eta) = (\rho|\hbar - \eta|, \sigma|\vartheta - \eta|)$$

where $\rho, \sigma > 0$, and $\hbar, \vartheta, \eta \in \mathfrak{R}$. Then

$$f\hbar = \hbar^2 + 5$$

and

$$K\hbar = 2\hbar^2.$$

We have

$$S(f\hbar, f\vartheta, f\eta) = (\rho|\hbar^2 - \eta^2|, \sigma|\vartheta^2 - \eta^2|) = \tfrac{1}{2}S(K\hbar, K\vartheta, K\eta) \leq qS(K\hbar, K\vartheta, K\eta)$$

for $q \in [\frac{1}{2}, 1)$, $f(\mathfrak{R}) = [5, \infty) \subseteq [0, \infty) = K(\mathfrak{R})$ and a self map K is continuous on $\mathfrak{R}$ and $f(\mathfrak{R})$ is V-complete subspace of $\mathfrak{R}$. Hence K and f have common fixed point that is unique.

Example 2.6 Let $V = \mathbb{R}$ with coordinatewise ordering and $\mathfrak{R} = [0,1]$

$$S(\hbar, \vartheta, \eta) = |\hbar - \eta| + |\vartheta - \eta|$$

where $\hbar, \vartheta, \eta \in \mathfrak{R}$. Then

$$f\hbar = \frac{\hbar}{4}$$

and

$$K\hbar = \frac{\hbar}{2}.$$

We have

$$S(f\hbar, f\vartheta, f\eta) = |\frac{\hbar}{4} - \frac{\eta}{4}| + |\frac{\vartheta}{4} - \frac{\eta}{4}| = \frac{1}{2}S(K\hbar, K\vartheta, K\eta) \leq qS(K\hbar, K\vartheta, K\eta)$$

for $q \in [\frac{1}{2}, 1)$, $f(\mathfrak{R}) = [0, \frac{1}{4}] \subseteq [0, \frac{1}{2}] = K(\mathfrak{R})$ and a self map K is continuous on $\mathfrak{R}$ and $f(\mathfrak{R})$ is V-complete subspace of $\mathfrak{R}$. Hence K and f have common fixed point that is unique.

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