Some fixed point theorems for self-mappings onvector valued rectangular metric spaces

Krishan Kumar
Department of Mathematics, Indira Gandhi University Meerpur, Rewari, Haryana, India
Mamta Kamra
Department of Mathematics, Indira Gandhi University Meerpur, Rewari, Haryana, India
Rajpal
Department of Mathematics, Raffles University, Neemrana, Alwar, Rajasthan, India

Abstract
Rectangular metric space was first defined by Branciari [3] in 2000. In this paper we prove some fixed point theorems on vector valued rectangular metric space. Our results generalize some fixed point results for scalar valued.

Keywords: Fixed point, Contraction Principle, Riesz space, Vector metric space, Rectangular metric space, Vector valued rectangular metric space.

1. Introduction and Preliminaries

Cuneyt Cevıĸ and Ishaĸ Altun [4] introduced the concept of vector metric space. Vector metric space is a generalization of metric space where metric is Riesz space valued. Branciari [3] definedthe notion of rectangular metric space and established some fixed point theorems for this space. By combining vector metric space and rectangular metric space, we introduce vector valued rectangular metric space and prove some fixed point results for this space.
We follow the concepts and terminology by C. D. Aliprantis and K. C. Border [1] for Riesz spaces.A lattice is a partially ordered set in which every couple of elements has both a least upper bound and a greatest lower bound. A partially ordered vector space $\boldsymbol{Q}$ is a Riesz space if it is also a lattice under its ordering. In Riesz space $\boldsymbol{Q}$, let $\varsigma \in \boldsymbol{Q}$ then the positive part ς^+, the negative part ς^- and the absolute value $|\varsigma|$ are defined below,

$$\varsigma^+ = \varsigma \vee 0,\ \ \varsigma^- = (-\varsigma) \vee 0 \text{ and } |\varsigma| = \varsigma \vee (-\varsigma).$$

Let $\varsigma_m \in \boldsymbol{Q}$, then the notation $\varsigma_m \downarrow y$ means that $\{\varsigma_m\}$ is a decreasing sequence in Riesz space $\boldsymbol{Q}$ such that $\inf \varsigma_m = \varsigma$. A Riesz space $\boldsymbol{Q}$ is an Archimedean if $\frac{1}{m}\varsigma \downarrow 0$ for every $\varsigma \in \boldsymbol{Q}_+$ where $\boldsymbol{Q}_+ = \{\varsigma \in \boldsymbol{Q} : 0 \preccurlyeq \varsigma\}$.

Example 1.1. Let we consider a non-empty topological space $\boldsymbol{W}$ and a real vector space $C(\boldsymbol{W})$ ofall real continuous functions on $\boldsymbol{W}$, then $C(\boldsymbol{W})$ is a Riesz space w.r.t. the partial ordering defineas if $f \preccurlyeq g$, where $f, g \in C(\boldsymbol{W})$ implies $f(\varsigma) \preccurlyeq g(\varsigma)$ for every $\varsigma \in \boldsymbol{W}$.
Lemma 1.2. Let $\boldsymbol{Q}$ be a Riesz space and $\varsigma \preccurlyeq k\varsigma$, $\forall\, \varsigma \in \boldsymbol{Q}_+$ also $k \in [0,1)$ then $\varsigma = 0$.
Definition 1.3. Let $\boldsymbol{Q}$ be a Riesz space and $\varsigma_m \in \boldsymbol{Q}$ then the sequence $\{\varsigma_m\}$ is said to be order convergent to ς written as $\varsigma_m \xrightarrow{o} \varsigma$ $\exists$ sequence $\{b_m\}$ in $\mathbf{Q}$ such that $b_m \downarrow 0$ as well as $\forall$ m,
$|\varsigma_m - \varsigma| \preccurlyeq b_m$.
The sequence $\{\varsigma_m\}$ is called order-Cauchy if $\exists$ sequence $\{b_m\}$ in $\boldsymbol{Q}$ such that $b_m \downarrow 0$ as well as
$|\varsigma_m - \varsigma_{m+p}| \preccurlyeq b_m \ \forall\, m$ and p.
Definition 1.4. Let $\boldsymbol{Q}$ be a Riesz space and $\boldsymbol{W}$ be a non-empty set. Then vector metric is a mapping $\kappa : \boldsymbol{W} \times \boldsymbol{W} \to \boldsymbol{Q}$ which satisfy all the properties given below:

(a) $\kappa(\varsigma, \xi) = 0 \Leftrightarrow \varsigma = \xi$

(b) $\kappa(\varsigma, \xi) = \kappa(\xi, \varsigma)$

(c) $\kappa(\varsigma, \xi) \preccurlyeq \kappa(\varsigma, \eta) + \kappa(\eta, \xi) \quad \forall\, \eta, \varsigma, \xi \in \boldsymbol{W}$.

And the triplet $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is called vector metric space(we will denote this space as VMS).

Example 1.5. Let $\boldsymbol{Q}$ be a Riesz space. The mapping $\kappa : \boldsymbol{Q} \times \boldsymbol{Q} \to \boldsymbol{Q}$ defined as:

$$\kappa(\varsigma, \xi) = |\varsigma - \xi| \qquad \forall\, \varsigma, \xi \in \boldsymbol{Q}.$$

Then κ is a vector metric and this vector metric is called absolute valued metric on $\boldsymbol{Q}$.

Example 1.6. Let $\mathbf{R}^2 = \boldsymbol{Q}$ and define a function $\kappa : \mathbf{R} \times \mathbf{R} \to \boldsymbol{Q}$ s.t.

$$\kappa(\varsigma, \xi) = (\beta_1|\varsigma - \xi|, \beta_2|\varsigma - \xi|)$$

where $0 < \beta_1 + \beta_2$ and $0 \le \beta_1, \beta_2$. Then the function κ is a vector metric with coordinatewise ordering and $(\mathbf{R}, \kappa, \boldsymbol{Q})$ is a VMS.

Definition 1.7. A sequence $\{\varsigma_m\}$ in a VMS $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is called vectorial convergent($\boldsymbol{Q}$-convergent)

to some $\varsigma \in \boldsymbol{W}$, written as $\varsigma_m \xrightarrow{\kappa, Q} \varsigma$ if $\exists$ a sequence b_m in $\boldsymbol{Q}$ such that $b_m \downarrow 0$ as well as $\kappa(\varsigma_m, \varsigma) \preccurlyeq b_m \ \forall\, m.$

Definition 1.8. A sequence $\{\varsigma_m\}$ in a VMS $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is called $\boldsymbol{Q}$-Cauchy if we get a sequence b_m in $\boldsymbol{Q}$ satisfying $b_m \downarrow 0$ as well as $\kappa(\varsigma_m, \varsigma_{m+p}) \preccurlyeq b_m \ \forall\, m, p.$

Definition 1.9. A VMS $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is said to be $\boldsymbol{Q}$-complete if every $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$ is $\boldsymbol{Q}$-convergent to a limit in $\boldsymbol{W}$.

Definition 1.10. Let $\boldsymbol{W}$ be no-empty set and the mapping κ: $\boldsymbol{W} \times \boldsymbol{W} \to$ R s.t. $\forall\, \varsigma, \xi \in \boldsymbol{W}$ and for all distinct points $\eta, \vartheta \in \boldsymbol{W}$ with $\eta, \vartheta \notin \{\varsigma, \xi\}$ satisfies all the properties given below:

(a) $\kappa(\varsigma, \xi) = 0 \Leftrightarrow \varsigma = \xi$

(b) $\kappa(\varsigma, \xi) = \kappa(\xi, \varsigma)$

(c) $\kappa(\varsigma, \xi) \le \kappa(\varsigma, \eta) + \kappa(\eta, \vartheta) + \kappa(\vartheta, \xi)$.

Then κ is called rectangular metric and the space $(\boldsymbol{W}, \kappa)$ is called a rectangular metric space(we denote this space as RMS). Below we give an example of a RMS but not a metric space.

Example 1.11. Let $\boldsymbol{W} = \mathbf{R}$, $t \in (0, \infty)$ and a mapping $\kappa : \boldsymbol{W} \times \boldsymbol{W} \to \mathbf{R}$ defined as

$$\kappa(\varsigma, \xi) = \begin{cases} 0 & if\ \varsigma = \xi \\ 3t & if\ (\varsigma, \xi) = (1,2)\ or\ (2,1),\ \ \varsigma \neq \xi \\ t & if\ (\varsigma, \xi) \neq (1,2)\ or\ (2,1),\ \ \varsigma \neq \xi \end{cases}$$

Then the space $(\boldsymbol{W}, \kappa)$ is a RMS. But $(\boldsymbol{W}, \kappa)$ is not metric space since

$$3t = \kappa(1, 2) > \kappa(1, 3) + \kappa(3, 2) = t + t.$$

Definition 1.12. Let $\boldsymbol{W}$ be non-empty set, $\boldsymbol{Q}$ be a Riesz space and the mapping κ: $\boldsymbol{W} \times \boldsymbol{W} \to \mathbf{Q}$ s.t. $\forall\, \varsigma, \xi \in \boldsymbol{W}$ and for all distinct points $\eta, \vartheta \in \boldsymbol{W}$ with $\eta, \vartheta \notin \{\varsigma, \xi\}$ satisfies all the properties given below:

(a) $\kappa(\varsigma, \xi) = 0 \Leftrightarrow \varsigma = \xi$

(b) $\kappa(\varsigma, \xi) = \kappa(\xi, \varsigma)$

(c) $\kappa(\varsigma, \xi) \preccurlyeq \kappa(\varsigma, \eta) + \kappa(\eta, \vartheta) + \kappa(\vartheta, \xi)$.

Then κ is called vector valued rectangular metric and the triplet $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is called vector valued RMS. Here we give an example of vector valued RMS but not vector metric space.

Example 1.13. Let $\boldsymbol{W} = \{0, 1, 2, 3\}$, $\boldsymbol{Q} = \mathbf{R}2$ and define $\kappa : \boldsymbol{W} \times \boldsymbol{W} \to \boldsymbol{Q}$ such that for all $\varsigma, \xi \in \boldsymbol{W}$, $\kappa(\varsigma, \xi) = \kappa(\xi, \varsigma)$ and

$$\kappa(\varsigma,\xi)=\begin{cases}(0,0) & if\ \varsigma=\xi\\ (3,3) & if\ \varsigma=3\ and\ \xi=2,\\ (1,1) & if\ \varsigma\in\{0,1\} and\ \xi\in\{1,2,3\}.\end{cases}$$

Here the space $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is vector valued RMS. But $\kappa(2, 0) + \kappa(0, 3) = (1, 1) + (1, 1) = (2, 2) \prec (3, 3) = \kappa(2, 3)$, so $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is not vector metric space.

2. Main Results

Motivated by the worк of Kamra et al.[7], Reich[9] and George et al.[5], we establish some fixedpoint results on vector valued RMS.

Lemma 2.1. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS. and $\boldsymbol{Q}$-Archimedean. Let $\{\varsigma_m\}$ be

a sequence in $\boldsymbol{W}$ such that

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \gamma\kappa(\varsigma_{m-1}, \varsigma_m) \qquad \forall\, m \in \mathbf{N}, \qquad (1)$$ where $0 \leq \gamma < 1$.

Then $\{\varsigma_m\}$ is $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$.

Proof. By (1), we have

$$\begin{aligned}\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \gamma\kappa(\varsigma_{m-1}, \varsigma_m) &\preccurlyeq \gamma^2\kappa(\varsigma_{m-2}, \varsigma_{m-1})\\ &\preccurlyeq \cdots\\ &\preccurlyeq \gamma^m\kappa(\varsigma_0, \varsigma_1). \qquad (2)\end{aligned}$$

And,

$$\begin{aligned}\kappa(\varsigma_m, \varsigma_{m+2}) \preccurlyeq \gamma\kappa(\varsigma_{m-1}, \varsigma_{m+1}) &\preccurlyeq \gamma^2\kappa(\varsigma_{m-2}, \varsigma_m)\\ &\preccurlyeq \cdots\\ &\preccurlyeq \gamma^m\kappa(\varsigma_0, \varsigma_2) \qquad (3)\end{aligned}$$

Now for the sequence $\{\varsigma_m\}$ we have $\kappa(\varsigma_m, \varsigma_{m+p})$ in two cases where $m, p \in \mathbf{N}$.

Case 1. If p is odd say $2n + 1$, $n \in \mathbf{N} \cup \{0\}$, then by (1) and the rectangle inequality

$$\begin{aligned}\kappa(\varsigma_m, \varsigma_{m+p}) &\preccurlyeq \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, \varsigma_{m+2}) + \kappa(\varsigma_{m+2}, \varsigma_{m+p})\\ &\preccurlyeq \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, \varsigma_{m+2}) + \kappa(\varsigma_{m+2}, \varsigma_{m+3}) + \kappa(\varsigma_{m+3}, \varsigma_{m+4}) + \kappa(\varsigma_{m+4}, \varsigma_{m+p})\\ &\preccurlyeq \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, \varsigma_{m+2}) + \cdots + \kappa(\varsigma_{m+2n-1}, \varsigma_{m+2n}) + \kappa(\varsigma_{m+2n}, \varsigma_{m+p})\\ &\preccurlyeq \gamma^m\kappa(\varsigma_0,\varsigma_1) + \gamma^{m+1}\kappa(\varsigma_0,\varsigma_1) + \cdots + \gamma^{m+2n}\kappa(\varsigma_0,\varsigma_1)\\ &\preccurlyeq \left\{\frac{\gamma^m}{1-\gamma}\kappa(\varsigma_0,\varsigma_1)\right\} \downarrow 0.\end{aligned}$$

Case 2. If p is even say $2n$, $1 \leq n \in \mathbf{N}$, then by (1), (2), (3) and the rectangle inequality

$$\begin{aligned}\kappa(\varsigma_m, \varsigma_{m+p}) &\preccurlyeq \kappa(\varsigma_m, \varsigma_{m+2}) + \kappa(\varsigma_{m+2}, \varsigma_{m+3}) + \kappa(\varsigma_{m+3}, \varsigma_{m+p})\\ &\preccurlyeq \kappa(\varsigma_m, \varsigma_{m+2}) + \kappa(\varsigma_{m+2}, \varsigma_{m+3}) + \kappa(\varsigma_{m+3}, \varsigma_{m+4}) + \kappa(\varsigma_{m+4}, \varsigma_{m+5}) + \kappa(\varsigma_{m+5}, \varsigma_{m+p})\\ &\preccurlyeq \kappa(\varsigma_m, \varsigma_{m+2}) + \kappa(\varsigma_{m+2}, \varsigma_{m+3}) + \cdots + \kappa(\varsigma_{m+2n-1}, \varsigma_{m+2n})\\ &\preccurlyeq \gamma^m\kappa(\varsigma_0, \varsigma_2) + \{\gamma^{m+2} + \gamma^{m+3} + \cdots + \gamma^{m+2n-1}\}\kappa(\varsigma_0, \varsigma_1)\\ &\preccurlyeq \left\{\gamma^m\kappa(\varsigma_0,\varsigma_2) + \frac{\gamma^{m+2}}{1-\gamma}\kappa(\varsigma_0,\varsigma_2)\right\} \downarrow 0.\end{aligned}$$

This implies that $\{\varsigma_m\}$ is $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$.

Theorem 2.2 Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\kappa(T\varsigma, T\vartheta) \preccurlyeq \gamma H(\varsigma, \vartheta)\ \forall\, \varsigma, \vartheta \in \boldsymbol{W},$$

where $\gamma \in [0, 1)$ and $H(\varsigma, \vartheta) \in \{\kappa(\varsigma, \vartheta), \kappa(\varsigma, T\varsigma), \kappa(\vartheta, T\vartheta), \kappa(\vartheta, T\varsigma)\}$. Then mapping T has a unique fixed point.

Proof. Let $\varsigma_0 \in \boldsymbol{W}$, and let $\varsigma_m = T\varsigma_{m-1}$, $m \in \mathbf{N}$. Then

$$\kappa(\varsigma_m, \varsigma_{m+1}) = \kappa(T\varsigma_{m-1}, T\varsigma_m) \preccurlyeq \gamma H(\varsigma_{m-1}, \varsigma_m)$$

Where

$$\begin{aligned} H(\varsigma_{m-1}, \varsigma_m) &\in \{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_{m-1}, T\varsigma_{m-1}), \kappa(\varsigma_m, T\varsigma_m), \kappa(\varsigma_m, T\varsigma_{m-1})\} \\ &= \{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(\varsigma_m, \varsigma_m)\} \\ &= \{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1}), 0\} \end{aligned}$$

The following three cases arise:

(i) $\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \gamma\kappa(\varsigma_{m-1}, \varsigma_m)$

(ii) $\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \gamma\kappa(\varsigma_m, \varsigma_{m+1})$ which gives $\kappa(\varsigma_m, \varsigma_{m+1}) = 0$.

(iii) $\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq 0$.

Thus

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \gamma\kappa(\varsigma_{m-1}, \varsigma_m) \qquad (4)$$

where $0 \le \gamma < 1$, then from above lemma 2.1, $\{\varsigma_m\}$ is $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$. Here $\boldsymbol{W}$ is $\boldsymbol{Q}$-complete, so $\exists$ some $s \in \boldsymbol{W}$ s.t. $\varsigma_m \xrightarrow{\kappa,\boldsymbol{Q}} s$. Then $\exists$ a sequence $\{b_m\} \in \boldsymbol{Q}$ s.t. $b_m \downarrow 0$ and $\kappa(\varsigma_m, s) \preccurlyeq b_m$. By repeating the process of (4), we get

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \gamma^m\kappa(\varsigma_0, \varsigma_1). \qquad (5)$$

We shall now show that s is a fixed point of T. For this

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq \kappa(s, \varsigma_m) + \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, Ts) \\ &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \kappa(T\varsigma_m, Ts) \\ &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma H(\varsigma_m, s) \end{aligned}$$

where $H(\varsigma_m, s) \in \{\kappa(\varsigma_m, s), \kappa(\varsigma_m, T\varsigma_m), \kappa(s, Ts), \kappa(s, T\varsigma_m)\}$.

We have the following four cases:

Case 1. If $H(\varsigma_m, s) = \kappa(\varsigma_m, s)$. Then

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma\kappa(\varsigma_m, s) \\ &\preccurlyeq \{ (1+\gamma)b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1)\} \downarrow 0 \end{aligned}$$

where $b_m \downarrow 0$ and $\gamma < 1$.

Case 2. If $H(\varsigma_m, s) = \kappa(\varsigma_m, T\varsigma_m)$. Then

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma\kappa(\varsigma_m, T\varsigma_m) \\ &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma^{m+1}\kappa(\varsigma_0, \varsigma_1) \\ &\preccurlyeq \left\{ b_m + \frac{\gamma^m}{1-\gamma}\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0 \end{aligned}$$

Case 3. If $H(\varsigma_m, s) = \kappa(s, Ts))$. Then

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma\kappa(s, Ts) \\ &\preccurlyeq \left\{\frac{1}{1-\gamma}b_m + \frac{\gamma^m}{1-\gamma}\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0 \end{aligned}$$

Case 4. If $H(\varsigma_m, s) = \kappa(s, T\varsigma_m)$. Then

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma\kappa(s, T\varsigma_m) \\ &= b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma\kappa(s, \varsigma_{m+1}) \\ &\preccurlyeq b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1) + \gamma b_{m+1} \\ &\preccurlyeq \{ (1+\gamma)b_m + \gamma^m\kappa(\varsigma_0, \varsigma_1)\} \downarrow 0 \qquad (\because b_{m+1} \preccurlyeq b_m) \end{aligned}$$

This implies $\kappa(s, Ts) = 0$ so $Ts = s$. Hence T has a fixed point. To prove the uniqueness of s, let if possible u is another fixed point of T. Then $Tu = u$ we get

$$\kappa(s, u) = \kappa(Ts, Tu) \preccurlyeq \gamma H(s, u)$$

where

$$\begin{aligned} H(s, u) &\in \{\kappa(s, u), \kappa(s, Tu), \kappa(u, Tu), \kappa(r, Tu)\} \\ &= \{\kappa(s, u), 0\} \end{aligned}$$

This implies $\kappa(s, u) = 0$ and so $s = u$. Hence T has fixed point which is unique.

Theorem 2.3. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and the mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\kappa(T\varsigma, T\vartheta) \preccurlyeq \gamma H(\varsigma, \vartheta) \ \forall\ \varsigma, \vartheta \in \boldsymbol{W},$$

where $\gamma \in [0, 1)$ and

$$H(\varsigma, \vartheta) \in \{\tfrac{1}{2}(\kappa(\varsigma,T\varsigma)+ \kappa(\vartheta,T\vartheta)), \tfrac{1}{2}(\kappa(\vartheta,T\varsigma)+ \kappa(\vartheta,T\vartheta)), \tfrac{1}{2}(\kappa(\vartheta, T\varsigma)+ \kappa(\varsigma, T\varsigma))\}.$$

Then mapping T has a unique fixed point.

Proof. Let $\varsigma_0 \in \boldsymbol{W}$, and let $\varsigma_m = T\varsigma_{m-1}$, $m \in \mathbf{N}$. Then

$$\kappa(\varsigma_m, \varsigma_{m+1}) = \kappa(T\varsigma_{m-1}, T\varsigma_m) \preccurlyeq \gamma H(\varsigma_{m-1}, \varsigma_m)$$

where

$$\begin{aligned} H(\varsigma_{m-1},\varsigma_m) \in\ & \{\tfrac{1}{2}(\kappa(\varsigma_{m-1},T\varsigma_{m-1})+ \kappa(\varsigma_m,T\varsigma_m)), \tfrac{1}{2}(\kappa(\varsigma_m,T\varsigma_{m-1})+ \kappa(\varsigma_m,T\varsigma_m)),\\ & \tfrac{1}{2}(\kappa(\varsigma_m, T\varsigma_{m-1})+ \kappa(\varsigma_{m-1}, T\varsigma_{m-1}))\}\\ =\ & \{\tfrac{1}{2}(\kappa(\varsigma_{m-1},\varsigma_m)+ \kappa(\varsigma_m,\varsigma_{m+1})), \tfrac{1}{2}(\kappa(\varsigma_m, \varsigma_m)+ \kappa(\varsigma_m, \varsigma_{m+1})),\\ & \tfrac{1}{2}(\kappa(\varsigma_m, \varsigma_m)+ \kappa(\varsigma_{m-1}, \varsigma_m))\}\\ =\ & \{\tfrac{1}{2}(\kappa(\varsigma_{m-1},\varsigma_m)+ \kappa(\varsigma_m,\varsigma_{m+1})), \tfrac{1}{2}\kappa(\varsigma_m, \varsigma_{m+1}), \tfrac{1}{2}\kappa(\varsigma_{m-1}, \varsigma_m)\}. \end{aligned}$$

The following three cases arise:

(i) $\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \frac{\gamma}{2}\,(\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\varsigma_m, \varsigma_{m+1}))$
implies

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \frac{\gamma}{2-\gamma}\kappa(\varsigma_{m-1}, \varsigma_m) \quad \text{where } \frac{\gamma}{2-\gamma} < 1.$$

(ii) $\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \frac{\gamma}{2}\kappa(\varsigma_m, \varsigma_{m+1})$ which gives $\kappa(\varsigma_m, \varsigma_{m+1}) = 0$.

(iii) $\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \frac{\gamma}{2}\kappa(\varsigma_{m-1}, \varsigma_m)$.

Thus

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \lambda\kappa(\varsigma_{m-1}, \varsigma_m) \quad \text{where } \lambda \in \left\{\frac{\gamma}{2}, \frac{\gamma}{2-\gamma}\right\} < 1 \text{ since } \gamma < 1. \qquad (6)$$

where $0 \le \gamma < 1$, then from above lemma 2.1, $\{\varsigma_m\}$ is $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$. Here $\boldsymbol{W}$ is $\boldsymbol{Q}$-complete, so $\exists$ some $s \in \boldsymbol{W}$ s.t. $\varsigma_m \xrightarrow{\kappa,\boldsymbol{Q}} s$. Then $\exists$ a sequence $\{b_m\} \in \boldsymbol{Q}$ s.t. $b_m \downarrow 0$ and $\kappa(\varsigma_m, s) \preccurlyeq b_m$. By repeating the process of (6), we get

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \lambda^m\kappa(\varsigma_0, \varsigma_1).$$

We shall now show that s is a fixed point of T. For this, we have

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq \kappa(s, \varsigma_m) + \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, Ts)\\ &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \kappa(T\varsigma_m, Ts)\\ &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \gamma H(\varsigma_m, s) \end{aligned}$$

where $H(\varsigma_m, s) \in \{\frac{1}{2}(\kappa(\varsigma_m,T\varsigma_m)+ \kappa(s,Ts)), \frac{1}{2}(\kappa(s,T\varsigma_m)+ \kappa(s,Ts)), \frac{1}{2}(\kappa(s, T\varsigma_m)+ \kappa(\varsigma_m,T\varsigma_m))\}$.

We have the following three cases:

Case 1. If $H(\varsigma_m, s) = \frac{1}{2}(\kappa(\varsigma_m, T\varsigma_m) + \kappa(s, Ts))$.

Then

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{\gamma}{2}(\kappa(\varsigma_m, T\varsigma_m) + \kappa(s, Ts))\\ &\preccurlyeq \frac{2}{2-\gamma}b_m + \frac{2}{2-\gamma}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{\gamma}{2-\gamma}\lambda^m\kappa(\varsigma_0, \varsigma_1)\\ &= \left\{\frac{2}{2-\gamma}b_m + \frac{2+\gamma}{2-\gamma}\lambda^m\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0 \end{aligned}$$

where $b_m \downarrow 0$ and $\lambda^m\kappa(\varsigma_0, \varsigma_1) \downarrow 0$.

Case 2. If $H(\varsigma_m, s) = \frac{1}{2}(\kappa(s,T\varsigma_m)+ \kappa(s,Ts))$. Then

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{\gamma}{2}(\kappa(s,T\varsigma_m)+ \kappa(s,Ts))\\ &= \frac{2}{2-\gamma}b_m + \frac{2}{2-\gamma}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{\gamma}{2-\gamma}\kappa(s,\varsigma_{m+1}) \end{aligned}$$

$$\preccurlyeq \frac{2}{2-\gamma} b_m + \frac{2}{2-\gamma}\lambda^m \kappa(\varsigma_0, \varsigma_1) + \frac{\gamma}{2-\gamma} b_{m+1}$$
$$\preccurlyeq \left\{\frac{2+\gamma}{2-\gamma}\, b_m + \frac{2}{2-\gamma}\lambda^m \kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0$$

Case 3. If $H(\varsigma_m, s) = \frac{1}{2}(\kappa(s, T\varsigma_m) + \kappa(\varsigma_m, T\varsigma_m))$. Then

$$\begin{aligned}\kappa(s, Ts) &\preccurlyeq b_m + \lambda^m \kappa(\varsigma_0, \varsigma_1) + \tfrac{\gamma}{2}(\kappa(s, T\varsigma_m) + \kappa(\varsigma_m, T\varsigma_m))\\ &\preccurlyeq b_m + \lambda^m \kappa(\varsigma_0, \varsigma_1) + \tfrac{\gamma}{2}\kappa(s, \varsigma_{m+1}) + \tfrac{\gamma}{2}\kappa(\varsigma_m, \varsigma_{m+1})\\ &\preccurlyeq b_m + \lambda^m \kappa(\varsigma_0, \varsigma_1) + \tfrac{\gamma}{2} b_{m+1} + \tfrac{\gamma}{2}\lambda^m \kappa(\varsigma_0, \varsigma_1)\\ &\preccurlyeq b_m + \tfrac{2+\gamma}{2}\lambda^m \kappa(\varsigma_0, \varsigma_1) + \tfrac{\gamma}{2} b_m\\ &= \left\{\frac{2+\gamma}{2} b_m + \frac{2+\gamma}{2}\lambda^m \kappa(\varsigma_0, \varsigma_1\right\}\\ &= \frac{2+\gamma}{2}\{ b_m + \lambda^m \kappa(\varsigma_0, \varsigma_1\} \downarrow 0.\end{aligned}$$

This implies $\kappa(s, Ts) = 0$ so $Ts = s$. Hence T has a fixed point s.
Next we shall prove the uniqueness of$s$. Let if possible, assume that u is another fixed point of
T. Implies $Tu = u$ we get

$$\kappa(s,u) = \kappa(Ts, Tu) \preccurlyeq \gamma H(s,u)$$

where

$$\begin{aligned}H(s, u) &\in \{\tfrac{1}{2}(\kappa(s,Ts) + \kappa(u,Tu)), \tfrac{1}{2}(\kappa(u,Ts) + \kappa(u,Tu)), \tfrac{1}{2}(\kappa(u, Ts) + \kappa(s, Ts))\}\\ &= \left\{\tfrac{1}{2}\kappa(s,u), 0\right\}\end{aligned}$$

This implies $\kappa(s, u) = 0$ and so $s = u$. Hence T has a unique fixed point.

Theorem 2.4. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\begin{aligned}\kappa(T\varsigma, T\vartheta) \preccurlyeq\ & a_1 \max\{\kappa(\varsigma, \vartheta), \kappa(\varsigma, T\varsigma), \kappa(\vartheta, T\vartheta)\} + a_2\{\kappa(\varsigma, \vartheta) + \kappa(\vartheta, T\varsigma)\}\\ &+ a_3\{\kappa(\varsigma, T\varsigma) + \kappa(\vartheta, T\varsigma)\} + a_4\{\kappa(\vartheta, T\vartheta) + \kappa(\vartheta, T\varsigma)\} \quad \forall\, \varsigma, \vartheta \in \boldsymbol{W}\end{aligned}$$

where $0 < a_1, a_2, a_3, a_4$ and $a_1 + 2a_2 + a_3 + a_4 < 1$. Then T has a unique fixed point.

Proof. Let $\varsigma_0 \in \boldsymbol{W}$ and let $\varsigma_m = T\varsigma_{m-1}$, $m \in \mathbf{N}$. Then

$$\begin{aligned}\kappa(\varsigma_m, \varsigma_{m+1}) &= \kappa(T\varsigma_{m-1}, T\varsigma_m)\\ &\preccurlyeq a_1 \max\{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_{m-1}, T\varsigma_{m-1}), \kappa(\varsigma_m, T\varsigma_m)\} + a_2\{\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\varsigma_m, T\varsigma_{m-1})\}\\ &\quad + a_3\{\kappa(\varsigma_{m-1}, T\varsigma_{m-1}) + \kappa(\varsigma_m, T\varsigma_{m-1})\} + a_4\{\kappa(\varsigma_m, T\varsigma_m) + \kappa(\varsigma_m, T\varsigma_{m-1})\}\\ &= a_1 \max\{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1})\} + a_2\{\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\varsigma_m, \varsigma_m)\}\\ &\quad + a_3\{\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\varsigma_m, \varsigma_m)\} + a_4\{\kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_m, \varsigma_m)\}\\ &= a_1 \max\{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1})\} + a_2\kappa(\varsigma_{m-1}, \varsigma_m) + a_3\kappa(\varsigma_{m-1}, \varsigma_m) + a_4\kappa(\varsigma_m, \varsigma_{m+1})\\ &= a_1 \max\{ \kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1})\} + (a_2 + a_3)\kappa(\varsigma_{m-1}, \varsigma_m) + a_4\kappa(\varsigma_m, \varsigma_{m+1}). \qquad (7)\end{aligned}$$

The following two cases arise:

Case 1. If $\max\{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1})\} = \kappa(\varsigma_{m-1}, \varsigma_m)$.
Then (7) implies

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq a_1\kappa(\varsigma_{m-1}, \varsigma_m) + (a_2 + a_3)\kappa(\varsigma_{m-1}, \varsigma_m) + a_4\kappa(\varsigma_m, \varsigma_{m+1})$$

which gives

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \frac{a_1+a_2+a_3}{1-a_4}\kappa(\varsigma_{m-1}, \varsigma_m) \quad \text{where } \frac{a_1+a_2+a_3}{1-a_4} < 1.$$

Case 2. If $\max\{\kappa(\varsigma_{m-1}, \varsigma_m), \kappa(\varsigma_m, \varsigma_{m+1})\} = \kappa(\varsigma_m, \varsigma_{m+1})$.
Then (7) implies

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq a_1\kappa(\varsigma_m, \varsigma_{m+1}) + (a_2 + a_3)\kappa(\varsigma_{m-1}, \varsigma_m) + a_4\kappa(\varsigma_m, \varsigma_{m+1})$$

which gives

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \frac{a_2+a_3}{1-a_1-a_4}\kappa(\varsigma_{m-1}, \varsigma_m) \quad \text{where } \frac{a_2+a_3}{1-a_1-a_4} < 1.$$

From above two cases, we have

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \lambda\kappa(\varsigma_{m-1}, \varsigma_m) \text{ where } \lambda \in \left\{\frac{a_1+a_2+a_3}{1-a_4}, \frac{a_2+a_3}{1-a_1-a_4}\right\} < 1. \qquad (8)$$

where $0 \leq \gamma < 1$, then from lemma 2.1, $\{\varsigma_m\}$ is $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$. Here $\boldsymbol{W}$ is $\boldsymbol{Q}$-complete, so ∃ some $s \in \boldsymbol{W}$ s.t. $\varsigma_m \xrightarrow{\kappa,\boldsymbol{Q}} s$. Then ∃ a sequence $\{b_m\} \in \boldsymbol{Q}$ s.t. $b_m \downarrow 0 \in \boldsymbol{Q}$ s.t. $b_m \downarrow 0$ and $\kappa(\varsigma_m, s) \preccurlyeq b_m$. By repeating the process of (8), we get

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \lambda^m\kappa(\varsigma_0, \varsigma_1). \qquad (9)$$

We shall now show that s is a fixed point of T. Now

$$\begin{aligned}
\kappa(s, Ts) &\preccurlyeq \kappa(s, \varsigma_m) + \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, Ts)\\
&\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \kappa(T\varsigma_m, Ts)\\
&\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + a_1\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, T\varsigma_m), \kappa(s, Ts)\} + a_2\{\kappa(\varsigma_m, s) + \kappa(s, T\varsigma_m)\}\\
&\quad + a_3\{\kappa(\varsigma_m, T\varsigma_m) + \kappa(s, T\varsigma_m)\} + a_4\{\kappa(s, Ts) + \kappa(s, T\varsigma_m)\}\\
&= b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + a_1\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(s, Ts)\} + a_2\{\kappa(\varsigma_m, s) + \kappa(s, \varsigma_{m+1})\}\\
&\quad + a_3\{\kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(s, \varsigma_{m+1})\} + a_4\{\kappa(s, Ts) + \kappa(s, \varsigma_{m+1})\}\\
&\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + a_1\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(s, Ts)\} + a_2(b_m + b_{m+1})\\
&\quad + a_3\lambda^m\kappa(\varsigma_0, \varsigma_1) + a_3 b_{m+1} + a_4\kappa(s, Ts) + a_4 b_{m+1}\\
&\preccurlyeq \frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{a_1}{1-a_4}\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(s, Ts)
\end{aligned}$$

We have the following three cases:

Case 1. If $\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(s, Ts)\} = \kappa(\varsigma_m, s)$. Then, we have

$$\begin{aligned}
\kappa(s, Ts) &\preccurlyeq \frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{a_1}{1-a_4}\kappa(\varsigma_m, s)\\
&\preccurlyeq \left\{\frac{1+a_1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0
\end{aligned}$$

where $b_m \downarrow 0$ and $\lambda^m\,\kappa(\varsigma_0, \varsigma_1) \downarrow 0$

Case 2. If $\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(s, Ts)\} = \kappa(\varsigma_m, \varsigma_{m+1})$.
Then

$$\begin{aligned}
\kappa(s, Ts) &\preccurlyeq \frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{a_1}{1-a_4}\kappa(\varsigma_m, \varsigma_{m+1})\\
&\preccurlyeq \frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{a_1}{1-a_4}\kappa(\varsigma_0, \varsigma_1)\\
&= \left\{\frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3+a_1}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0
\end{aligned}$$

Case 3. If $\max\{\kappa(\varsigma_m, s), \kappa(\varsigma_m, \varsigma_{m+1}), \kappa(s, Ts)\} = \kappa(s, Ts)$. Then

$$\begin{aligned}
\kappa(s, Ts) &\preccurlyeq \frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1) + \frac{a_1}{1-a_4}\kappa(s, Ts)\\
&= \left\{\frac{1+2a_2+a_3+a_4}{1-a_4} b_m + \frac{1+a_3}{1-a_1-a_4}\lambda^m\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0.
\end{aligned}$$

Then from above three cases, we have $\kappa(s, Ts) = 0$ so $Ts = s$. Hence s is a fixed point of T.
Now we shall show that s is unique. Let, if possible, assume that u is another fixed point of T. Implies $Tu = u$ we get

$$\begin{aligned}
\kappa(s, u) &= \kappa(Ts, Tu)\\
&\preccurlyeq a_1\max\{\kappa(s, u), \kappa(s, Ts), \kappa(u, Tu)\} + a_2\{\kappa(s, u) + \kappa(u, Ts)\} + a_3\{\kappa(s, Ts) + \kappa(u, Ts)\}\\
&\quad + a_4\{\kappa(u, Tu) + \kappa(u, Ts)\}\\
&= a_1\kappa(s, u) + a_2\{\kappa(s, u) + \kappa(u, s)\} + a_3\kappa(u, Ts) + a_4\kappa(u, s)\\
&= (1 + 2a_2 + a_3 + a_4)\,\kappa(s, u).
\end{aligned}$$

Since $1 + 2a_2 + a_3 + a_4 < 1$, this implies $\kappa(s, u) = 0$ and so $s = u$. Hence T has a unique fixed point.

Theorem 2.5. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\kappa(T\varsigma, T\vartheta) \preccurlyeq \alpha_1 \kappa(\varsigma, \vartheta) + \alpha_2 \kappa(\varsigma, T\varsigma) + \alpha_3 \kappa(\vartheta, T\vartheta) \qquad \forall \varsigma, \vartheta \in \boldsymbol{W}$$

where $0 < \alpha_1, \alpha_2, \alpha_3$ with $\sum_{i=1}^{3} \alpha_i < 1$. Then T has a unique fixed point.

Proof. Let $\varsigma_0 \in \boldsymbol{W}$ and let $T\varsigma_{m-1}$, $m \in \mathbf{N}$. Then

$$\begin{aligned} \kappa(\varsigma_m, \varsigma_{m+1}) &= \kappa(T\varsigma_{m-1}, T\varsigma_m) \\ &\preccurlyeq \alpha_1 \kappa(\varsigma_{m-1}, \varsigma_m) + \alpha_2 \kappa(\varsigma_{m-1}, T\varsigma_{m-1}) + \alpha_3 \kappa(\varsigma_m, T\varsigma_m) \\ &= \alpha_1 \kappa(\varsigma_{m-1}, \varsigma_m) + \alpha_2 \kappa(\varsigma_{m-1}, \varsigma_m) + \alpha_3 \kappa(\varsigma_m, \varsigma_{m+1}) \\ &= \frac{\alpha_1+\alpha_2}{1-\alpha_3} \kappa(\varsigma_{m-1}, \varsigma_m) \end{aligned}$$

which implies

$$\kappa(\varsigma_{m-1}, \varsigma_m) \preccurlyeq \frac{\alpha_1+\alpha_2}{1-\alpha_3} \kappa(\varsigma_{m-1}, \varsigma_m) \qquad (10)$$

Also

$$\begin{aligned} \kappa(\varsigma_{m+1}, \varsigma_m) &= \kappa(T\varsigma_m, T\varsigma_{m-1}) \\ &\preccurlyeq \alpha_1 \kappa(\varsigma_m, \varsigma_{m-1}) + \alpha_2 \kappa(\varsigma_m, T\varsigma_m) + \alpha_3 \kappa(\varsigma_{m-1}, T\varsigma_{m-1}) \\ &= \alpha_1 \kappa(\varsigma_{m-1}, \varsigma_m) + \alpha_2 \kappa(\varsigma_m, \varsigma_{m+1}) + \alpha_3 \kappa(\varsigma_{m-1}, \varsigma_m) \\ &= \frac{\alpha_1+\alpha_3}{1-\alpha_2} \kappa(\varsigma_{m-1}, \varsigma_m) \end{aligned}$$

which implies

$$\kappa(\varsigma_{m+1}, \varsigma_m) \preccurlyeq \frac{\alpha_1+\alpha_3}{1-\alpha_2} \kappa(\varsigma_{m-1}, \varsigma_m) \qquad (11)$$

By combining (10) and (11), we get

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \lambda\kappa(\varsigma_{m-1}, \varsigma_m). \qquad (12)$$

where

$$\lambda = \max\left\{\frac{a_1+a_2}{1-a_3}, \frac{a_1+a_3}{1-a_2}\right\} < 1.$$

where $0 \leq \gamma < 1$, then from lemma 2.1, $\{\varsigma_m\}$ is $\boldsymbol{Q}$-Cauchy sequence in $\boldsymbol{W}$. Here $\boldsymbol{W}$ is $\boldsymbol{Q}$-complete, so $\exists$ some $s \in \boldsymbol{W}$ s.t. $\varsigma_m \xrightarrow{\kappa,\boldsymbol{Q}} s$. Then $\exists$ a sequence $\{b_m\} \in \boldsymbol{Q}$ s.t. $b_m \downarrow 0 \in \boldsymbol{Q}$ s.t. $b_m \downarrow 0$ and $\kappa(\varsigma_m, s) \preccurlyeq b_m$. By repeating the process of (12), we get

$$\kappa(\varsigma_m, \varsigma_{m+1}) \preccurlyeq \lambda^m\kappa(\varsigma_0, \varsigma_1).$$

We shall now show that s is a fixed point of T. For this, we have

$$\begin{aligned} \kappa(s, Ts) &\preccurlyeq \kappa(s, \varsigma_m) + \kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_{m+1}, Ts) \\ &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \kappa(T\varsigma_m, Ts) \\ &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \alpha_1 \kappa(\varsigma_m, s) + \alpha_2 \kappa(\varsigma_m, \varsigma_{m+1}) + \alpha_3 \kappa(s, Ts) \\ &\preccurlyeq b_m + \lambda^m\kappa(\varsigma_0, \varsigma_1) + \alpha_1 b_m + \lambda^m\alpha_2 \kappa(\varsigma_0, \varsigma_1) + \alpha_3 \kappa(s, Ts) \\ &= \left\{\frac{1+a_1}{1-a_3} b_m + \frac{1+a_2}{1-a_3}\lambda^m\kappa(\varsigma_0, \varsigma_1)\right\} \downarrow 0. \end{aligned}$$

This gives κ(s, Ts) = 0, so Ts = s. Hence T has a fixed point s.

We shall now show the uniqueness of s. Let, if possible, assume that u is another fixed point of T. Then $Tu = u$ we get

$$\begin{aligned} \kappa(s, u) &= \kappa(Ts, Tu) \\ &\preccurlyeq \alpha_1 \kappa(s, u) + \alpha_2 \kappa(s, Ts) + \alpha_3 \kappa(u, Tu) \\ &= \alpha_1 \kappa(s, u). \end{aligned}$$

Since $0 \leq \alpha_1 < 1$, this implies $\kappa(s, u) = 0$ and so $s = u$. Hence T has a unique fixed point.

Corollary 2.6. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\kappa(T\varsigma, T\vartheta) \preccurlyeq \gamma\kappa(\varsigma, \vartheta) \quad \forall \varsigma, \vartheta \in \boldsymbol{W},$$

where $\gamma \in [0, 1)$ Then T has a unique fixed point in $\boldsymbol{W}$.

This is Banach Contraction principle for vector valued rectangular metric space.

Proof. Result follows by taking $\alpha_1 = \gamma$, $\alpha_2 = \alpha_3 = 0$ in Thm. 2.5.

Corollary 2.7. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\kappa(T\varsigma, T\vartheta) \preccurlyeq \gamma(\kappa(T\varsigma, \varsigma) + \kappa(T\vartheta, \vartheta)) \qquad \forall\, \varsigma, \vartheta \in \boldsymbol{W},$$

where $\gamma \in [0, \frac{1}{2})$ Then T has a unique fixed point in $\boldsymbol{W}$.

This result is Kannan Type contraction for vector valued rectangular metric space.

Proof. Result follows by taкing $\alpha_2 = \alpha_3 = \gamma$, $\alpha_1 = 0$ in Thm. 2.5.

Corollary 2.8. Let $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ be a complete vector valued RMS with $\boldsymbol{Q}$-Archimedean and mapping $T : \boldsymbol{W} \to \boldsymbol{W}$ satisfies the contractive condition

$$\kappa(T\varsigma, T\vartheta) \preccurlyeq \gamma(\kappa(\varsigma, \vartheta) + (\varsigma, T\varsigma) + \kappa(\vartheta, T\vartheta)) \qquad \forall\, \varsigma, \vartheta \in \boldsymbol{W},$$

where $\gamma \in [0, \frac{1}{3})$ Then T has a unique fixed point in $\boldsymbol{W}$.

Proof. Result follows by taкing $\alpha_1 = \alpha_2 = \alpha_3 = \gamma$ in Thm. 2.5.

Examle 2.9. Let $\boldsymbol{W} = \{\varsigma \in \mathbf{N} : 2 \le \varsigma < 7\}$, $\boldsymbol{Q} = \mathbf{R}2$ and $\kappa : \boldsymbol{W} \times \boldsymbol{W} \to \boldsymbol{Q}$ be defined as:
$\kappa(\varsigma, \vartheta) = \kappa(\vartheta, \varsigma)$ and

$$\kappa(\varsigma,\vartheta) = \begin{cases} (0,0) \;\; if\ \varsigma = \vartheta, \\ (4,8)\ if\ \varsigma = 2\ and\ \vartheta = 3, \\ (2,3)\ if\ \varsigma = \{2,3\} and\ \vartheta = 4, \\ (1,3)\ if\ \varsigma = \{2,3,4\} and\ \vartheta = 5, \\ (4,6)\ if\ \varsigma = \{2,3,4,5\} and\ \vartheta = 6. \end{cases}$$

Then $(\boldsymbol{W}, \kappa, \boldsymbol{Q})$ is vector valued RMS. but not vector metric space since

$$(2, 6) = \kappa(2, 5) + \kappa(5, 3) \prec \kappa(2, 3) = (4, 8).$$

Now take a self mapping T on $\boldsymbol{W}$ which define as:

$$T\varsigma = \begin{cases} 4 \;\; if\ \varsigma \neq 6 \\ 3 \;\; if\ \varsigma = 6 \end{cases}$$

Then $\kappa(T2, T3) = \kappa(T2, T4) = \kappa(T2, T5) = \kappa(T3, T4) = \kappa(T3, T5) = \kappa(T4, T5) = (0, 0)$ and in all other cases

$$\kappa(T2, T6) = \kappa(T3, T6) = \kappa(T4, T6) = \kappa(T5, T6) = \kappa(4, 3) = (2, 3), \text{ and}$$
$$\kappa(2, 6) = \kappa(3, 6) = \kappa(4, 6) = \kappa(5, 6) = (4, 6).$$

Then for $\gamma \in \left[\frac{1}{2}, 1\right)$ shows that T satisfies the conditions of Cor. 2.6. and $\varsigma = 4$ is its unique fixed
point.

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